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SANTA CLARA VALLEY WATER DISTRICT

ADOBE CREEK PHASE IV

HYDRAULIC CAPACITY ANALYSIS

ALMA STREET TO CHARLESTON ROAD

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Preface

Homeowners adjacent to Adobe Creek between Alma Street and Charleston Road have questioned the need for the proposed Adobe Creek Phase IV project. They have asked the District to determine the existing channel capacity and describe the method used to calculate the channel capacity. This report addresses these concerns.

Summary

This report contains a description of the District's method of hydraulic capacity analysis as applied to Adobe Creek.

It includes, in Appendix 1, a summary of basic hydraulic definitions and the assumptions made in our analysis. It also includes, for additional background, excerpts from the textbook "Open Channel Hydraulics", by Ven Te Chow, 1959, in Appendix 2, considered to be the standard among open channel hydraulics references, and the "HEC-2 Water Surface Profiles Users Manual", 1982, in Appendix 3.

This analysis determined that the channel capacity is 1100 cubic feet per second (cfs) between Alma Street and Charleston Road.

Basic Hydraulic Principles

The basis of all hydraulics is: (1) the conservation of the flow's energy, consisting of the flow's depth and velocity, between channel sections, and (2) the continuity of flow in the channel (ie, the amount of flow does not change). In assessing the water surface profile in a channel, the most basic procedure is the comparison of specific energy between channel sections.

As explained in Appendix 1, P. 13, the specific energy for a channel section is defined as:

$E = Y + V^2/2g$ where E = specific energy (ft)
Y = depth of flow (ft)
V = velocity of flow (ft/sec) $V = Q/A$ where
  Q = flow amount (cfs)
  A = flow area (ft^2)
g = gravitational acceleration (ft^2/sec)

Figure 1, p. 4 shows an example with a channel plan, profile, and cross-sections which illustrates the variation in channel geometry along Adobe Creek. Figure 2, p. 5 shows a graph of specific energy vs. flow depth for both of these cross-sections.

At the upstream channel section A, the flow is at normal depth (Appendix 1, p. 18), Yn (where gravity forces balance the friction forces in the channel; where the slope of the channel energy, water surface profile, and the channel bottom are equal; and where the flow is in equilibrium), and this normal depth is below the minimum specific energy of the section. The specific energy at Y = Yn is En-A (see Figure 2).

As shown in Figure 2, the minimum specific energy, Emin-B, required in the downstream channel section B is greater than En-A. This means that the flow at section A cannot proceed downstream without additional energy.

The only available source of energy here is additional depth of flow. In order to pass through this transition, the flow will develop a hydraulic jump (Appendix 1, p. 16) from point 1 to 2 on Figure 2.

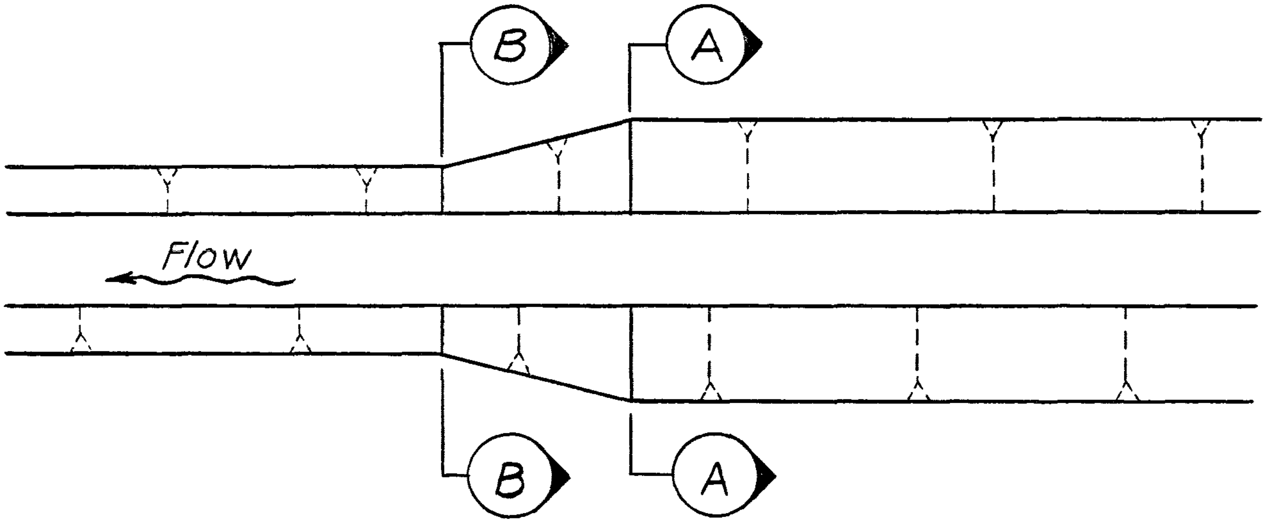
Still more energy is needed to pass through section B at the downstream end of the transition. The flow will rise up to provide the additional energy (from point 2 to point 3 on Figure 2). In the example, the water surface would rise nearly 8 feet before there is enough energy to flow into section B.

At section B, in order for the flow to return to normal depth in the channel, the flow will follow the path through the section's minimum energy from point 3 to point 5 in Figure 2. The water surface profile which finally results from this energy balancing is shown on Figure 1.

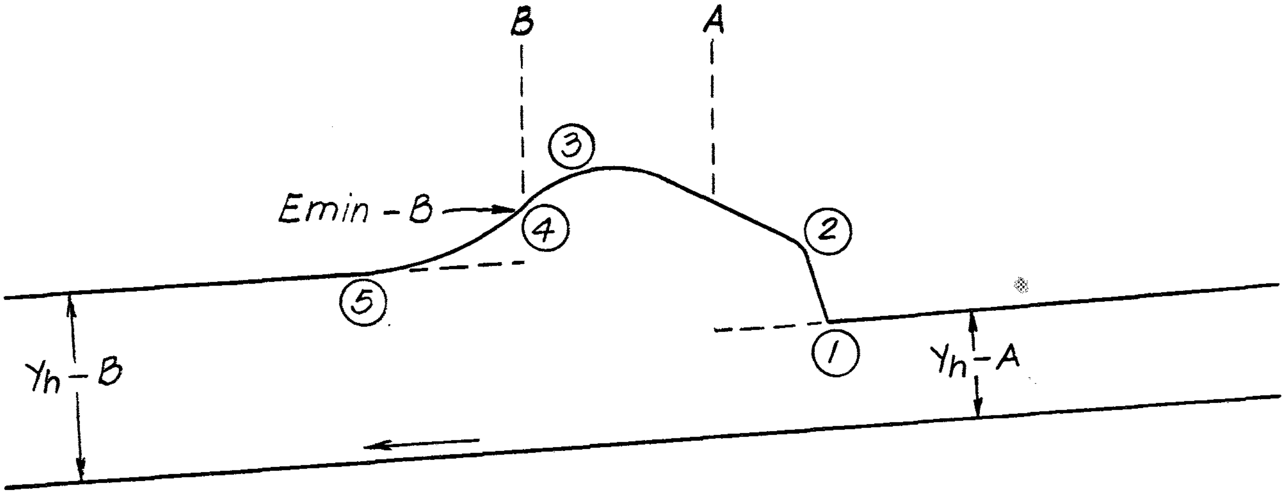
While not shown, these hydraulic jumps and transitions also have energy losses associated with them.

This whole process of the flow lacking enough energy to pass through a channel section, with subsequent hydraulic jump, is typical of the channel on Adobe Creek, and, indeed, the concepts utilized are basic to hydraulics.

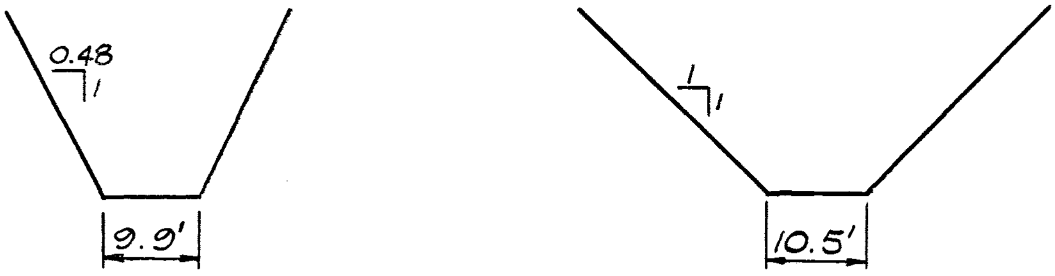
A description of the procedure used to calculate the water surface profiles and the channel capacity follows in this report.



PLAN



PROFILE



SECTION B

SECTION A

FIGURE 1

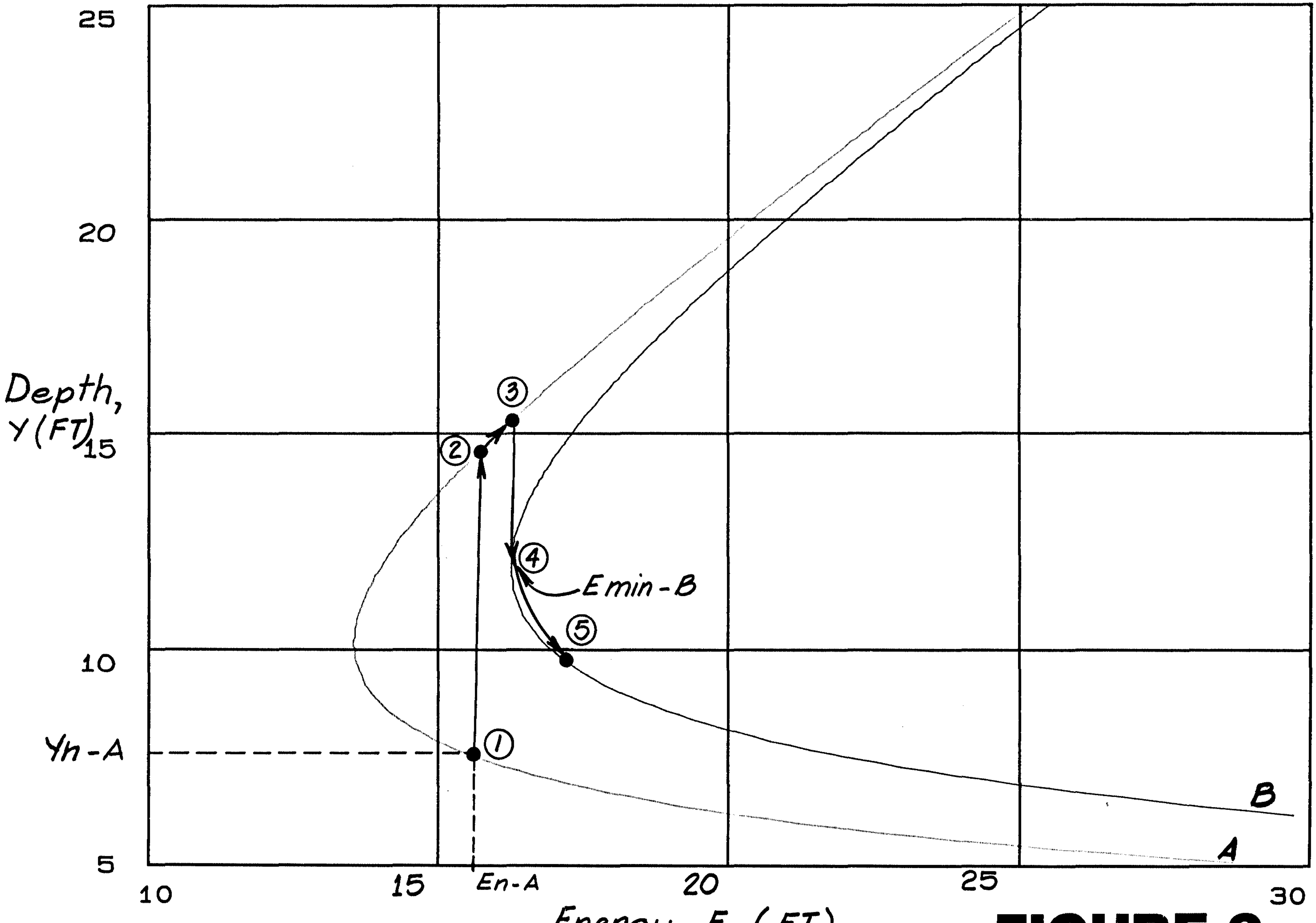

25
20
15
10
5
Depth, Y (FT)
Yn-A
10
15
En-A
20
25
30
Energy, E (FT)
①
②
③
④
⑤
Emin-B
A
B

FIGURE 2

Background

1985 Planning Study

Preliminary hydraulic analysis done by the District's PreDesign Division for the 1985 Adobe Creek Planning Study estimated the existing Adobe Creek channel to have a capacity of 1300 cfs (1985 Planning Study, Bibliography, Ref. # 4).

This 1985 analysis assumed a concrete lined trapezoidal channel section with a constant 10 foot bottom width and 0.75 : 1.00 side slopes. Since the channel was prismatic, there were no eddy losses (Appendix 1, p. 24) due to expansions or contractions in the channel. The other channel features, such as the periodic drops in the channel bottom, were based upon the 1957 channel construction plans. It was assumed that no superelevation (Appendix 1, p. 24) or crosswaves (Appendix 1, p. 25) due to channel curves were present.

1989 Analysis

After obtaining survey data in 1987, the District's Design Division observed that the channel cross-sections varied considerably. In the 66 cross-sections taken over approximately 2300 feet of channel, bottom widths varied from 9.6 feet to 10.75 feet and side slopes varied from 0.47:1.00 to 1.00:1.00.

This variation in the channel cross-section means that the channel is non-prismatic, and therefore eddy losses are a factor. With the eddy losses, the flow regime becomes susceptible to changes from supercritical to subcritical flow (Appendix 1, p. 15). The flow in the supercritical sections remained unstable (Appendix 1, p. 15).

The non-prismatic channel, repeated changes in flow regime from supercritical to subcritical flow, crosswaves and superelevation, and flow instability all produce a very complex flow profile.

Hydraulic Analysis Criteria and Assumptions

Due to the channel's non-prismatic nature, the standard-step method (Appendix 1, p. 21) must be used to analyze the water surface in the channel. This is an iterative procedure that balances energy, friction, and continuity of flow along a channel. Due to it's acceptance by FEMA (Appendix 1, p. 24) and other agencies, it's relative ease of use, and it's reliability, the District uses the HEC-2 (Appendix 3) program from the U.S. Corps of Engineer's Hydrologic Engineering Center to determine water surface profiles. Other procedures are used to supplement the HEC-2 results where appropriate.

The criteria and assumptions that have been used for analyzing the hydraulic capacity of Adobe Creek are:

1. flow is steady (ie, the quantity does not change with time);
2. flow is gradually-varying (Appendix 1, p. 21);
3. flow is one-dimensional;
4. channel slopes are "small" - ie, less than 10%;
5. the channel is non-prismatic (channel geometry is based on the 1987 survey data);
6. n = Manning's roughness coefficient. = .015 for a concrete channel (Appendix 1, p. 20);
7. the eddy loss coefficients for contractions and expansion are 0.1 and 0.3 (Appendix 1, p. 24), respectively, throughout this channel reach;
8. Flow is always contained within and above the channel banks.
9. Within channel curves, the water surface analysis must consider the effects of crosswaves and superelevation. For a trapezoidal channel, the following equations were applied to the HEC-2 output to obtain the height of crosswaves and superelevation (Appendix 1, p. 24 and p. 25):

$$hs = \frac{bV^2}{2gRc}$$

where hs = difference in water surface elevation (in feet) between mean, HEC-2, water surface and outer water surface due to superelevation
b = channel width at water surface (ft)
Rc = radius of bend (ft)
V = flow velocity (ft/sec)
g = gravitational acceleration (ft/sec^2)

hc = hs
= difference in water surface elevation (in feet) between mean, HEC-2, water surface and outer water surface due to crosswaves

hs and hc are added to the HEC-2 water surface to calculate the actual water surface within a curve. The crosswaves also continue downstream of the curve a "considerable" distance. We assume that the crosswaves dissipate linearly in 300 feet.

10. Finally, where HEC-2 indicates that flow could be either subcritical or supercritical, we compare the supercritical flow's sequent depths (Appendix 1, p. 15) to the subcritical flow depths to locate hydraulic jumps (Appendix 1, p. 16).

Procedure

1. Model the existing channel with HEC-2 between Charleston Road and Alma St. using the previously outlined design assumptions and criteria. Since experience has shown that the flow regime changes from supercritical to subcritical and back repeatedly, the channel is modelled for both supercritical and subcritical flow conditions.

2. Develop water surface profiles by varying flow rates between 700 and 3100 cfs. 700 cfs is an arbitrarily chosen flow rate thought to be below the capacity of the channel. 3100 cfs is the 1% design flow rate (obtained from "Adobe Creek Hydrology", Bibliography, Ref. # 6).

3. For each flow rate, produce a "composite" water surface consisting of the synthesis of the subcritical and supercritical water surface. The procedure for this is:

    a. Break the channel reach up into sections (generally from drop to drop in the channel bottom, and suspected to contain a hydraulic jump).

    b. Locate the control sections (Appendix 1, p. 20) where the flow passes through , critical depth. For supercritical flow, the control section is at the upstream end of the section. For subcritical flow, the control section is at the downstream end of the section.

    c. When the HEC-2 model for subcritical flow produces subcritical flow at a cross-section, and the model for supercritical flow at the same cross-section produces supercritical flow, there may be a hydraulic jump from supercritical flow to subcritical flow.

    HEC-2 does not handle hydraulic jumps, with their high internal energy losses.

    In a hydraulic jump, momentum is conserved. Since momentum is conserved, the specific force before and after the hydraulic jump must be the same. The sequent depth is the depth at which the specific force is the same as the specific force of the supercritical flow.

In this situation, there are three possible outcomes:

  i. If the sequent depth remains above the subcritical depth, flow is supercritical for the cross-section:

  ii. If the sequent depth passes through the subcritical depth, there is a hydraulic jump at the cross-section.

  iii. If the sequent depth remains below the subcritical depth, flow is subcritical for the cross-section;

d. After the location of any hydraulic jumps have been determined, the subcritical and supercritical water surfaces are synthesized to produce a composite water surface profile.

4. Add superelevation and crosswaves, if appropriate, to the water surface profile.

5. For each cross-section, the water surface profile (incuding crosswaves and superelevation) is compared to the existing top of bank. The flow rate for the water surface profile that equals, but does not exceed, the existing top of bank is the capacity for this channel cross-section.

6. The minimum capacity of any of the channel cross-sections is the capacity of the reach of channel between Alma Street and Charleston Road.

Results

As provided by the HEC-2 analysis, a series of water surface profiles were generated for flow rates ranging from 700 to 3100 cfs. Subcritical and supercritical HEC-2 runs were compared. By locating control sections and hydraulic jumps, a composite water surface was calculated for each flow rate. The effects of crosswaves and superelevation were added where appropriate.

The capacity of the existing channel between Alma Street and Charleston Road is 1100 cfs.

(1) Chow Ven Te, PHD, Open Channel Hydraulics, McGraw Hill Book Co., New York, 1959.

(2) Lenau, Charles W., ASCE, Supercritical Flow in Bends of

(3) L.A. County Flood Control District, Hydraulic Design Manual March, 1982. p. C-1 to C-18, F-1 to F-11.

(4) SCVWD, Adobe Creek Planning Study and Engineer's Report, June, 1985.

(5) U.S. Amy Corp of Engineers, The Hydrologic Engineering Center, HEC-2 Water Surface Profiles Users Manual, September, 1982

(6) A. Saah, SCVWD, Adobe Creek Hydrology, April, 1989.

APPENDIX 1

Basic Definitions and Assumptions

Energy

The total energy of water in any streamline passing through a channel section may be expressed as the total head of water, which is equal to the sum of the elevation above a datum, the pressure head, and the velocity head.

H = z + d*cos( )+v^2/2*g where

H = Total energy, or head, (ft)
z = elevation of channel bottom (ft)
d = depth of water (ft)
= slope of channel(degrees) v =
velocity of flow (ft/sec) g =
gravitational acceleration (ft/sec^2)

for a channel of large slope (see Figure 3, p. 14).

For a channel of small slope, H = z + d + v ^ 2 / 2 * g

By the principle of conservation of energy, the total energy head at the upstream section 1 equals the total energy head at the downstream section 2 plus the loss of energy He between the two sections. Z1 + Y1 + V1^2/2g = Z2 + Y2 + V2^2/2g + He

where He = energy losses due to friction and eddy losses

This formula is the well-known Bernoulli's Energy Equation, and is the basis of the standard step methods used for evaluating water surface profiles in non-prismatic channels. It is also used in dealing with transitions between prismatic channel sections.

Specific Energy

Specific energy is the energy per pound of water at any section of channel measured with respect to the channel bottom.

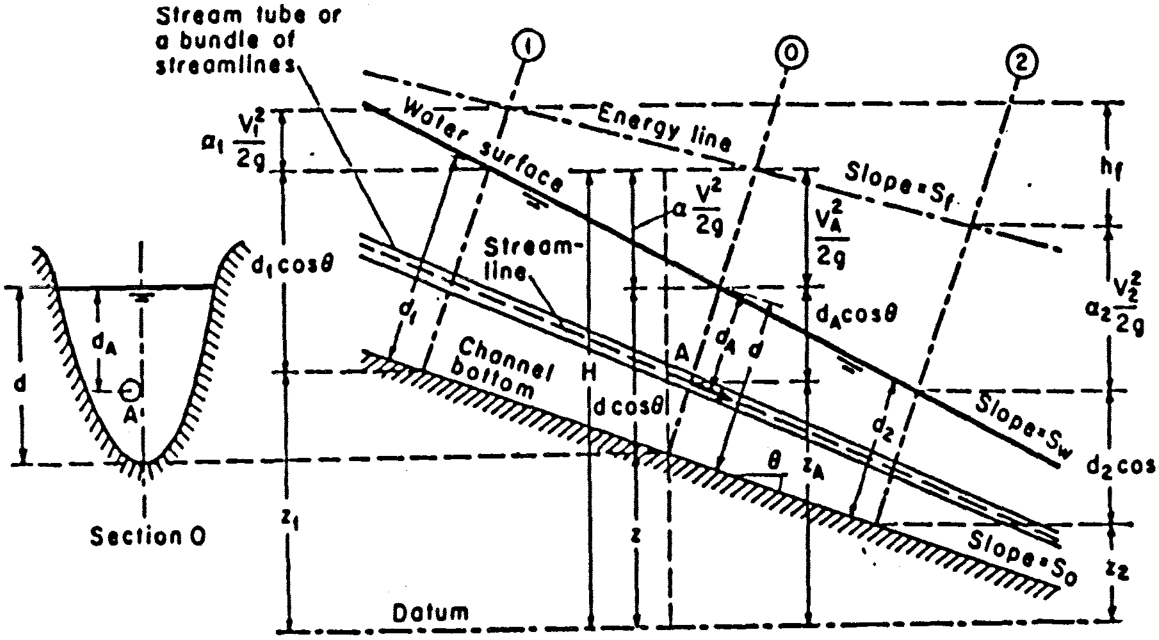
E = Y + V^2/2g where

E = specific energy (ft)
Y = depth (ft)
V = velocity (ft/sec)
g = acceleration due to gravity (ft/sec^2)

See Figure 4, p. 14 for a graph of E vs y.

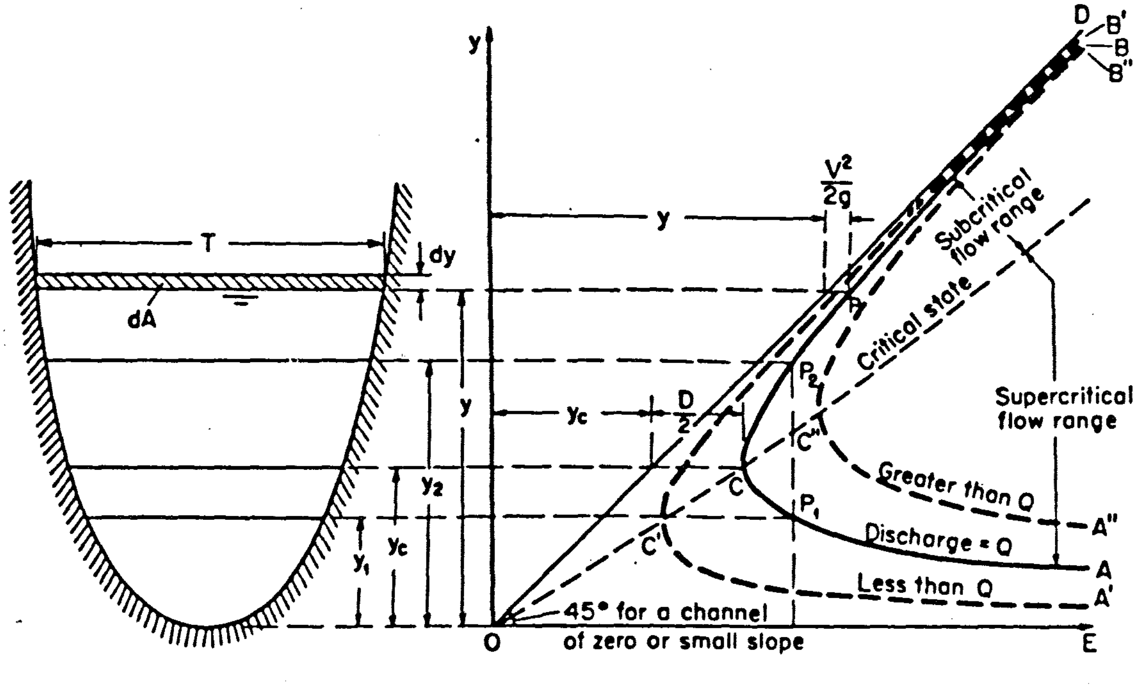
Critical Depth

Critical depth, Yc, occurs when the specific energy is at a minimum for a channel cross-section.



Energy in gradually varied open-channel flow.

FIGURE 3



Specific-energy curve.

FIGURE 4

Flow Instability

> Flow at or near the critical state is unstable. This is because a minor change in specific energy at or close to critical state will cause a major change in depth....It can be observed that, when the flow is near the critical state, the water surface appears unstable and wavy. Such phenomena are generally caused by minor changes in energy due to variations in channel roughness, cross section, channel slope, or deposits of sediment or debris. *In the design of a channel, if the depth is found at or near the critical depth for a great length of channel, the shape or slope of the channel should be altered . . . in order to secure greater stability.*
>
> Chow, 1959.

Critical slope is that slope that maintains flow at a uniform and critical depth. Flow at or near critical slope will be unstable.

The U.S. Corps of Engineers, the U.S. Soil Conservation Service (SCS), the L.A. County Flood Control District, and the District all recognize and design for the effects of flow instability. These effects are taken into account by either avoiding channel configurations that have flow instability (preferred), or by providing additional freeboard.

For prismatic channels, the District's criteria for flow instability is $0.7S_c < S_o < 1.3S_c$ (the SCS criteria).

For non-prismatic channels, the specific energy cannot be too close to the minimum specific energy at critical depth.

Supercritical and Subcritical Flow

When $y < Y_c$, flow is supercritical. The inertial forces of the flow are dominant over the gravitational forces.

When $y > Y_c$, flow is subcritical. The gravitational forces dominate the inertial forces of the flow.

Hydraulic Drop

A hydraulic drop is a rapid change from supercritical flow to subcritical flow, generally caused by a abrupt change in the channel slope or cross section. The flow must pass through the critical depth, Y_c, where E is at a minimum. A hydraulic drop is also a flow control section.

The free-overfall is a special case of the hydraulic drop; slopes of 0.1 or above are considered to act as free-overfalls.

Hydraulic Jump

A hydraulic jump is a abrupt change in the depth of flow, from supercritical flow to subcritical flow.

Hydraulic jumps commonly involve large internal energy loss through dissipation in the turbulent body of water in the jump. Momentum is conserved, but energy is not. Specific force is used to deal with hydraulic jumps.

Specific Force

The specific force is the sum of the momentum of the flow passing through the channel section per unit time per unit weight of water and the force per unit weight of water. Specific force consists of applying the momentum principle to a short horizontal reach of prismatic channel, which permits the external force of friction and the weight of the water to be ignored.

$F = Q^2/(g*A) + z*A$ where F = specific force
Q = flow (cfs)
g = gravitational acceleration (ft/sec^2) A =
area of flow (ft^2)
z = distance of centroid of area A to surface of flow (ft)

See Figure 5, p. 17 for F vs. y.

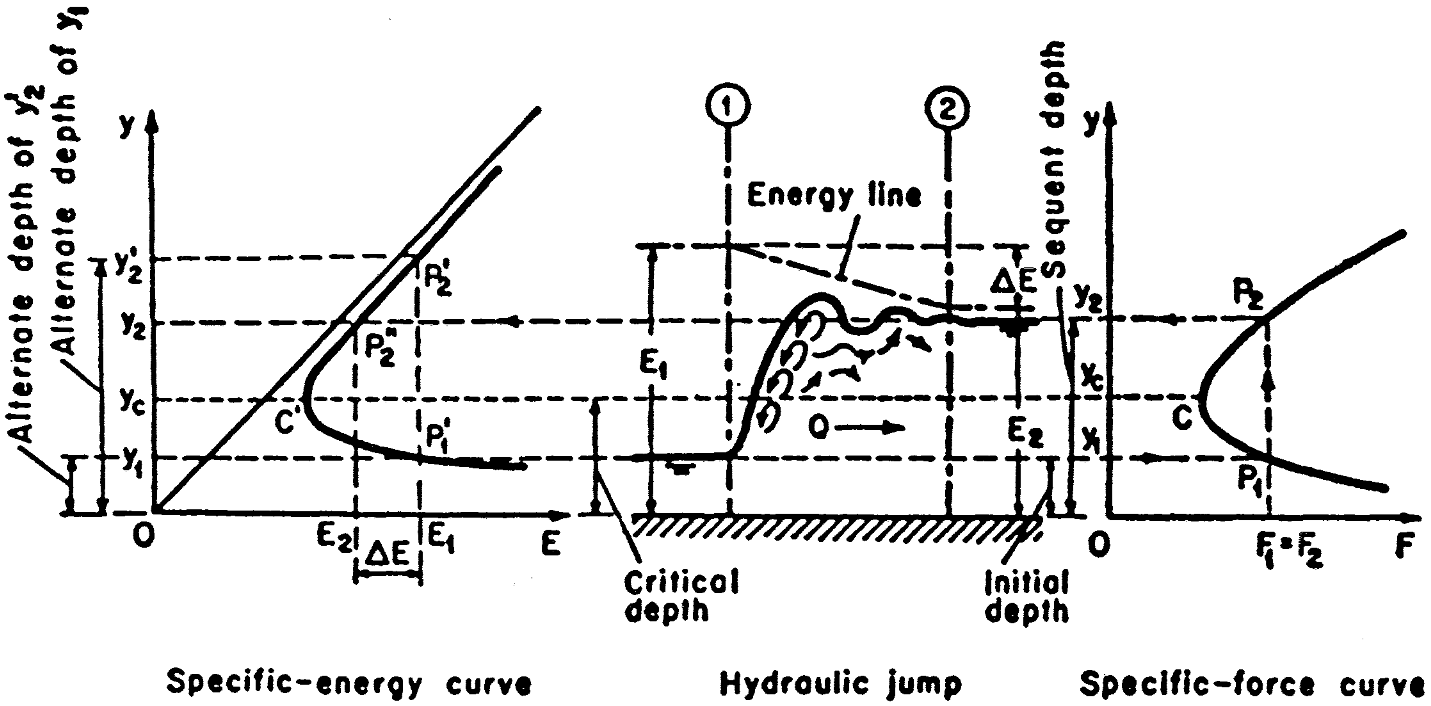
Specific force is used in situations where the high internal energy losses (such as occurs at hydraulic jumps) cannot be evaluated by energy methods.

Sequent Depth and Initial Depth

Initial depth is the depth of flow in a channel before a hydraulic jump. Sequent depth is the depth of flow in a channel after a hydraulic jump.

Since momentum is conserved in hydraulic jumps, the specific force at the initial depth and the sequent depth is the same.

A hydraulic jump may be located by comparing the sequent depth of the supercritical flow with the depth of subcritical flow.



Specific-energy curve Hydraulic jump Specific-force curve

Hydraulic jump interpreted by specific-energy and specific-force curves.

FIGURE 5

Uniform Flow and Normal Depth

Uniform flow is defined as flow with:

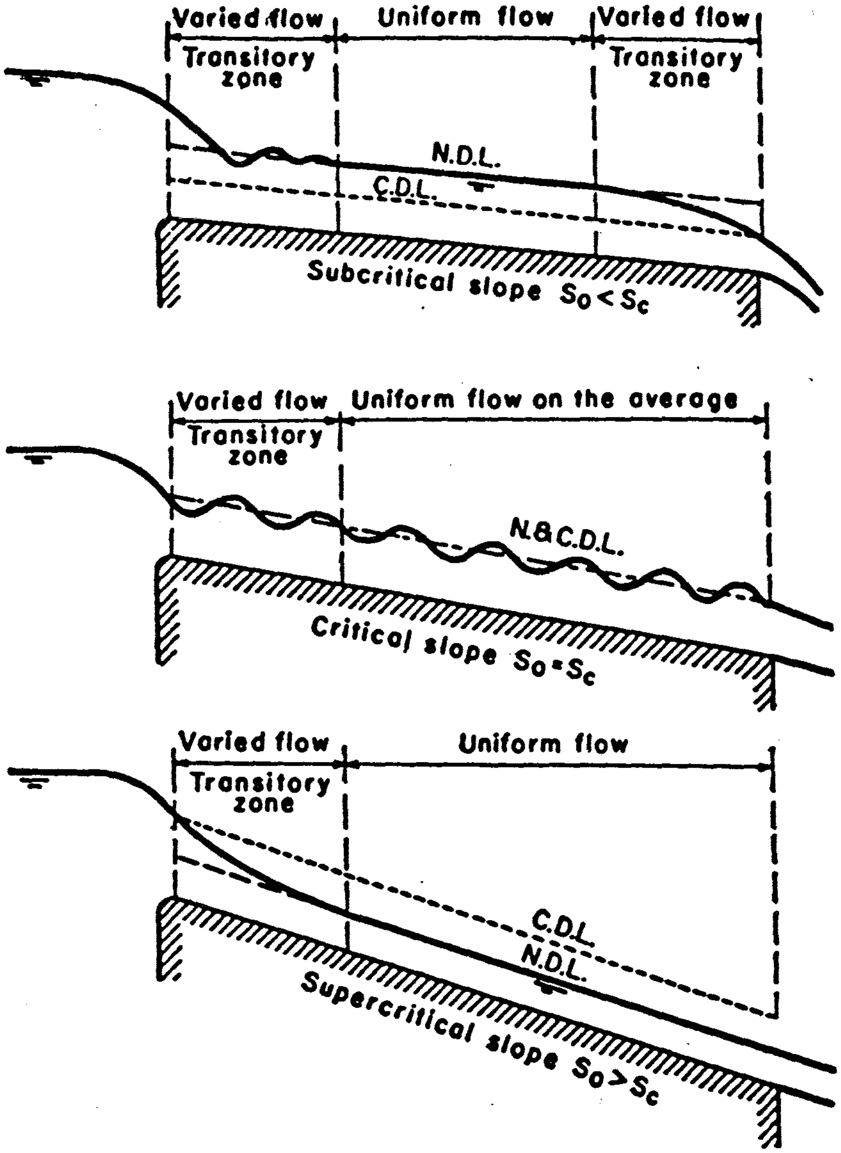
1) equal depth, water area, velocity (mean), and discharge at every section of the channel reach; and

2) parallel energy line, water surface, and channel bottom. Uniform flow will be developed when the resistance (friction) of a body of water is balanced by the gravity forces.

Normal depth is the unique depth for a channel with a given n, Q, and slope S for maintaining uniform flow.

See Figure 6, p. 19 for some examples of normal depth.

Normal depth only applies to prismatic channels.



Establishment of uniform flow in a long channel.

FIGURE 6

Manning's n Value

Manning's formula,

$$V = \frac{1.49}{n} * R^{(2/3)} * S^{(1/2)},$$

has become the most widely used of all uniform-flow formulas for open-channel flow computations. This is due to it's simplicity of form and the satisfactory results it lends to practical applications.

V = flow velocity
R = hydraulic radius = A / P, where A = flow area and
P = wetted perimeter
S = Channel slope
n = Manning's coefficient of roughness representing the channel roughness.

A transformation of this formula is used to calculate friction losses in both prismatic and non-prismatic channels (by the HEC-2 standard step method).

n values as used by different sources, for concrete channels:

Agency	n value
U.S. Dept. of Agriculture	.014
U.S. Corps of Engineers	.016
U.S. Soil Conservation Service	.015 (formed finish)
L.A. Co. Flood Control District	.014
Handbook of Hydraulics, King	.014 - .016
Open-Channel Hydraulics, Chow	.013 - .016 (float finish)
Santa Clara Valley Water District	.015

n as developed was intended for uniform flow. Experience has shown that it's use in the calculation water surface profiles with gradually-varied flow is adequate and reliable for analysis and design purposes.

Control Sections

The control of flow in an open channel means "the establishment of a definitive flow condition in the channel or, more specifically, a definitive relationship between the stage (depth) and the discharge (Q) of the flow." (Chow, Reference # 1, Bibliography) When the control of flow is achieved at a certain channel section, this section is a *control section.*

The control section controls the flow in such a way that it restricts the transmission of the effect of changes in flow condition either in an upstream direction or in a downstream condition depending on the state of flow in the channel *(ie, depending upon whether the flow is subcritical or supercritical).*

A critical flow section is always a control section.

Control sections in prismatic channels occur where the channel changes slope or cross-section.

Control sections in non-prismatic channels can only be identified as locations where the flow passes from the subcritical flow range through critical flow to the supercritical flow range.

The control of flow is said to be at the downstream end for channels with subcritical slope and at the upstream end for channels with supercritical slope.

Control sections are normally used as beginning water surfaces in determining water surface profiles.

See Figures 7 and 8, p. 22, for examples of control sections.

Gradually Varied Flow

Gradually varied flow is steady flow whose depth varies gradually along the length of the channel. This definition assumes: (1) the flow is steady (Q does not change); and (2) the pressure distribution over the channel cross-sections is hydrostatic.

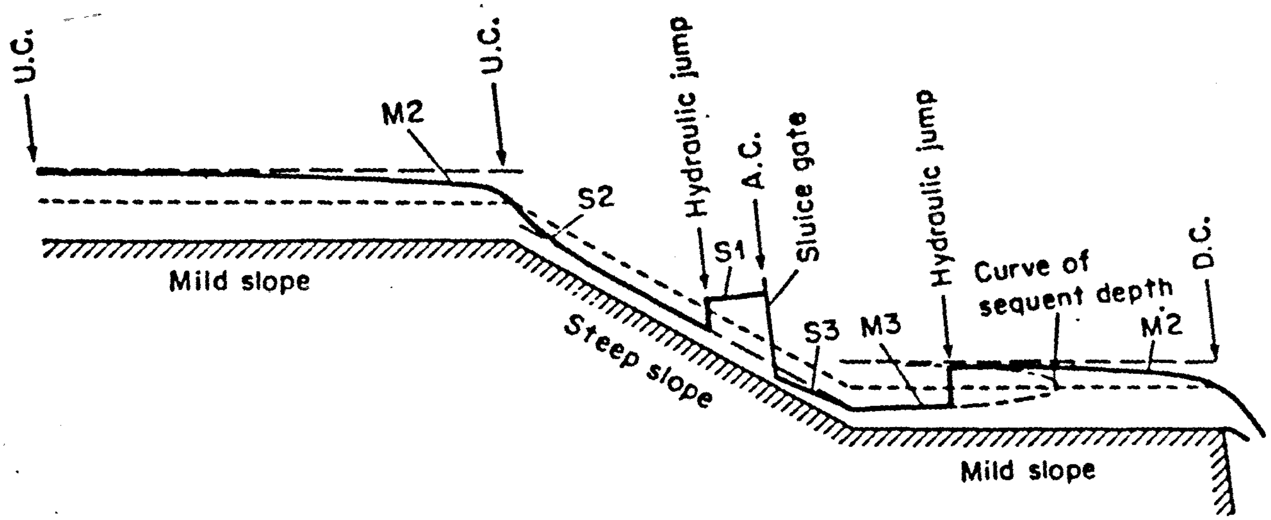
The other basic assumption for gradually varied flow is that: the head loss *(friction)* at a section is the same *(for gradually varied flow)* as for a uniform flow having the velocity and hydraulic radius of the section.

This assumption means that Manning's uniform-flow formula may be used to evaluate the friction losses between sections of a gradually varied flow, and that Manning's coefficient of roughness n is applicable to gradually varied flow.

Standard Step Method

The standard step method is used in computing the water surface profile when the channel is non-prismatic - ie, the cross-sections change. It is currently the only practical method for non-prismatic channels.

The computations are carried out, by steps, from cross-section to cross-section where the hydraulic characteristics have been determined by field survey.



Analysis of flow profile for Example 9-1. U.C. = upstream control; D.C. = downstream control; A.C. = artificial control.

FIGURE 7

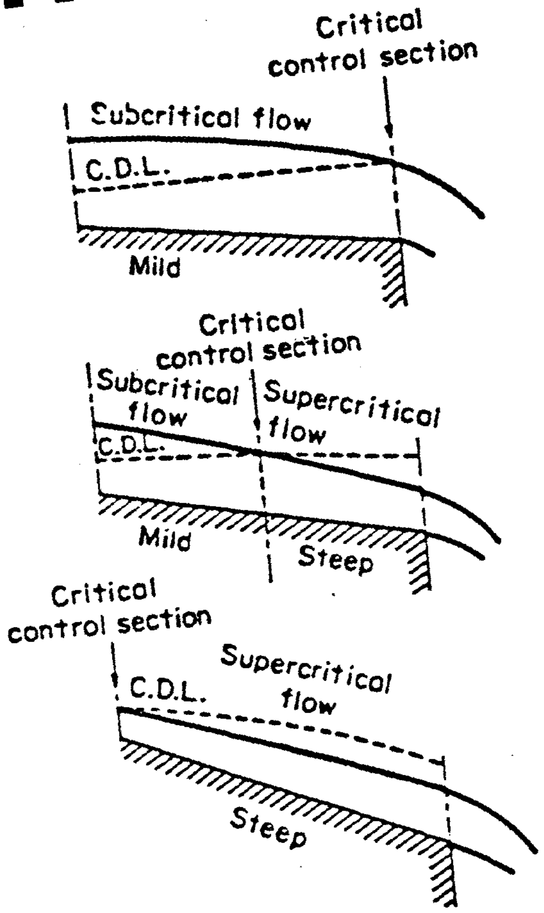


FIGURE 8

The steps of the method are:

1) A beginning water surface is assumed at the first cross-section.

2) Trial water surface elevations are assumed at the next cross-section until the energy and friction equations are satisfied;

3) Utilizing the water surface computed in (2), the process is repeated to the next section;

4) Step 3 is repeated to the end of the reach.

The equation to be solved is:

$$Z1 + \frac{V1^2}{2g} = Z2 + \frac{V2^2}{2g} + He$$

where Z1, Z2 = water surface elevations above datum
He = friction + eddy losses

The District uses the standard step program HEC-2, from the Corps of Engineers Hydrologic Engineering Center, to calculate water surface profiles for natural and non-prismatic channels.

Since it is an energy method, it cannot deal with situations with high internal energy losses - ie, hydraulic jumps. Momentum principles (such as specific force) should be used in those situations.

Calculations must be carried out upstream for subcritical flow and downstream for supercritical flow (this is because the calculations must proceed from a control section); subcritical flow is downstream control and supercritical flow is upstream control.

If the flow may be both in a reach, water surface profiles for both must be calculated and the results synthesized.

The standard step method also converges to the correct water surface provided that the computation is carried out in the correct direction. This is useful if a starting water surface elevation is not known.

FEMA

The Federal Emergency Management Agency (FEMA) is the agency which is responsible for the administration of the National Flood Insurance Program. FEMA is also responsible for the review and update of the Flood Insurance Rate Maps, which are the basis of determining the flood insurance requirements and rates. The District policy is to perform hydraulic analyses and designs in a manner acceptable to FEMA.

The HEC-2 program is accepted by FEMA.

Eddy Losses

Eddy losses occur due to changes in the channel cross section. For non-prismatic channels, or transitions between different prismatic channel sections, eddy losses may be appreciable.

Eddy losses depend mainly upon the velocity head change and the coefficient of contraction or expansion, k.

$$He = k * \frac{(V1^2 - V2^2)}{2g}$$

Different sources cite these k values:

Source	k	
	Contraction	Expansion
L.A. Co. Flood Control District	0.1	0.2
Handbook of Hydraulics, King	0.1	0.2
Open-Channel Hydraulics, Chow	0.1	0.2
U.S. Army Corps of Engineers	0.1	0.2
U.S. Army Corps of Engineers, Hydrologic Engineering Center	0.1	0.3
Santa Clara Valley Water District	0.1	0.3

In our capacity analysis, the application of these eddy loss coefficients to the non-prismatic channel had a dramatic effect. The losses prevented the flow from remaining supercritical as assumed in the 1985 Engineer's Report, and therefore reduced the channel capacity.

We performed a sensitivity analysis on these coefficients by using the typical factors divided by 10 - Ie, 0.01 and 0.03. The flow still jumped from supercritical to subcritical, as expected by a comparison of the typical specific energy curves in Figure 2, p. 5.

Superelevation

Superelevation is a rise in the water surface at the outer bank, and a drop in the water surface at the inner bank, in a channel cross-section at a bend. This is caused by the centrifugal force acting on the flow around the bend.

$$hs = \frac{bV^2}{2gRc}$$ where

hs = difference in water surface elevation between mean water surface and outer water surface due to superelevation
b = channel width at water surface
Rc = radius of bend V = flow velocity
g = gravitational acceleration

(Lenau, Bibliography, Reference # 2, and L.A. County Flood Control District, Reference # 3)

Crosswaves

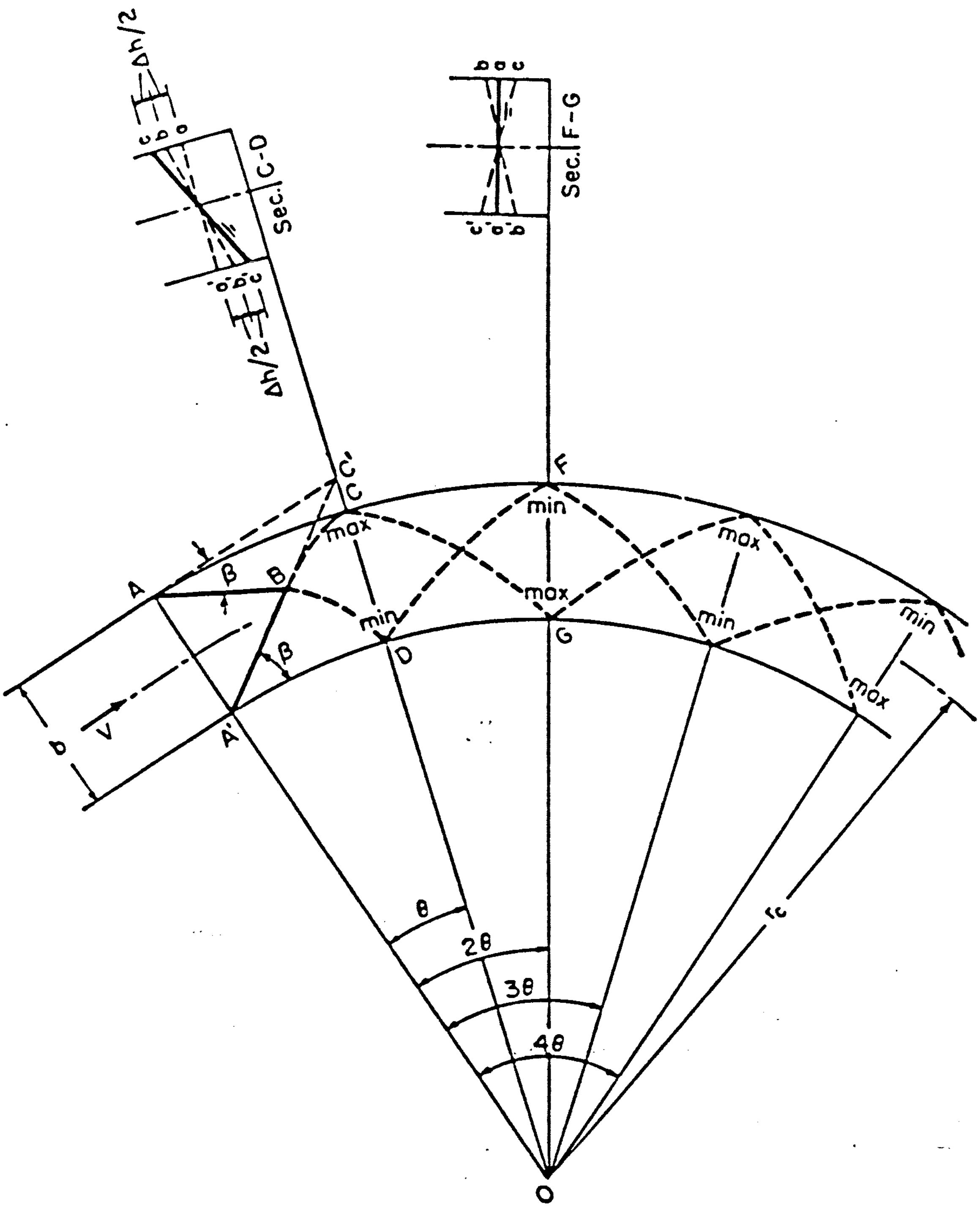
Crosswaves are found in supercritical flow in channels with non-linear alignment, or non-prismatic cross-sections. In a curved channel, they are caused by the turning effect of the curved wall, which does not act equally on all the channel section streamlines.

They form a disturbance pattern that can persist for a considerable distance downstream of the phenomena causing them.

Photographs taken in 1986 clearly show crosswaves in Adobe Creek.

$$hc = \frac{b*V^2}{2gRc}$$ = difference in water surface elevation between mean water surface and outer water surface due to crosswaves (equal in magnitude to hs for superelevation)

See figure 9, p. 26 for an example of crosswaves and superelevation.



Cross-wave pattern for supercritical flow in a curved channel.

FIGURE 9

APPENDIX 2

Excerpts from Open Channel Flow, by Chow

APPENDIX 3

Excerpts from HEC-2 manual

Computer Program 723-X6-L202A

HEC-2

Water Surface Profiles

Users Manual

September 1982

US Army Corps of Engineers
Water Resources Support Center

The Hydrologic Engineering Center
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Chapter 2

THEORETICAL BASIS FOR PROFILE CALCULATION

1. GENERAL

This section describes methodology used in HEC-2 for the calculation of water surface profiles. Topics discussed include equations used for basic profile calculation, cross section subdivision for determining conveyance and velocity distribution, friction loss evaluation, iterative procedure for solving the basic equations and critical depth determination. Computational methodology for calculating flow through bridges is presented in Appendix IV. Methodology used by HEC-2 to determine and evaluate flood plain encroachments is contained in Appendix II.

2. EQUATIONS FOR BASIC PROFILE CALCULATION

The following two equations are solved by an iterative procedure (the standard step method) to calculate an unknown water surface elevation at a cross section:

$$WS_2 + \frac{\alpha_2 V_2^2}{2g} = WS_1 + \frac{\alpha_1 V_1^2}{2g} + h_e \tag{1}$$

$$h_e = L\,\bar{S}_f + C \left| \frac{\alpha_2 V_2^2}{2g} - \frac{\alpha_1 V_1^2}{2g} \right| \tag{2}$$

where:

WS_1, WS_2 = water surface elevations at ends of reach (see Figure 1)

V_1, V_2 = mean velocities (total discharge ÷ total flow area) at ends of reach

α_1, α_2 = velocity coefficients for flow at ends of reach

g = acceleration of gravity

h_e = energy head loss

L = discharge-weighted reach length

$\bar{S}_f$ = representative friction slope for reach

C = expansion or contraction loss coefficient

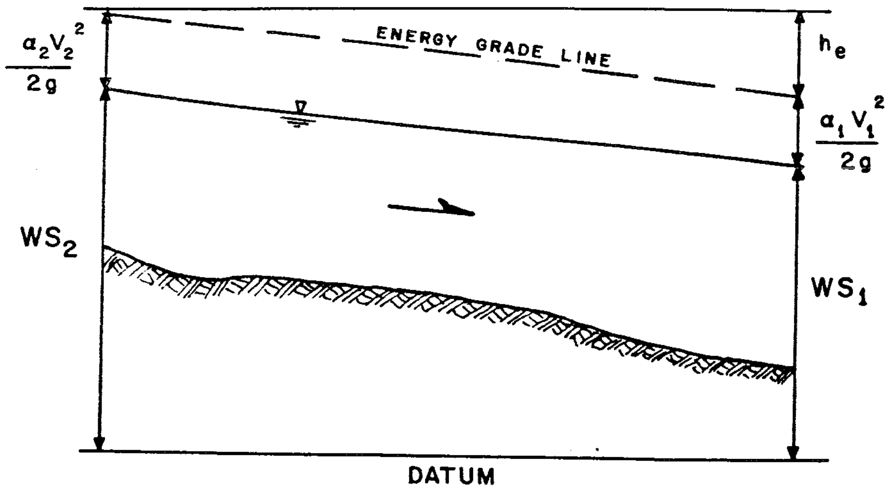

Figure 1. Representation of Terms in Energy Equation

The discharge-weighted reach length, L, is calculated as:

$$L = \frac{L_{lob}\bar{Q}_{lob} + L_{ch}\bar{Q}_{ch} + L_{rob}\bar{Q}_{rob}}{\bar{Q}_{lob} + \bar{Q}_{ch} + \bar{Q}_{rob}} \tag{3}$$

where:

L_{lob}, L_{ch}, L_{rob} = reach lengths specified for flow in the left overbank, main channel and right overbank, respectively

$\bar{Q}_{lob}$, $\bar{Q}_{ch}$, $\bar{Q}_{rob}$ = arithmetic average of flows at the ends of the reach for the left overbank, main channel, and right overbank, respectively

Determination of a representative friction slope, $\bar{S}_f$, is discussed in section 4 of this chapter. Selection of appropriate magnitudes for expansion and contraction coefficients is discussed in Chapter 3, Section 5 and Appendix IV.

3. CROSS SECTION SUBDIVISION

The determination of total conveyance and the velocity coefficient for a cross section requires that flow be subdivided into units for which the velocity is uniformly distributed. The approach used in HEC-2 is to subdivide flow in the overbank areas using the input cross section stati (X-coordinates) as the basis for subdivision. Conveyance is calculated within each subdivision by the equation (based on English units):

$$k = \frac{1.486}{n} a r^{2/3} \tag{4}$$

where:

k = conveyance for subdivision

n = Manning's 'n' for subdivision

a = flow area for subdivision

r = hydraulic radius for subdivision (area divided by wetted perimeter)

Flow in the main channel is not subdivided in normal applications. The total conveyance for the cross section is obtained by summing the incrementa conveyances.

The velocity coefficient, α, is obtained with the following equation:

$$\alpha = \frac{(A_t)^2 \left[(K_{lob})^3/(A_{lob})^2 + (K_{ch})^3/(A_{ch})^2 + (K_{rob})^3/(A_{rob})^2\right]}{(K_t)^3} \tag{5}$$

where:

A_t = total flow area of cross section

A_{lob}, A_{ch}, A_{rob} = flow areas of left overbank, main channel and right overbank, respectively

K_t = total conveyance of cross section

K_{lob}, K_{ch}, K_{rob} = conveyances of left overbank, main channel and right overbank, respectively

4. FRICTION LOSS EVALUATION

Friction loss is evaluated in HEC-2 as the product of $\bar{S}_f$, and L, where $\bar{S}_f$ is the representative friction slope for a reach and L is defined with equation (3). Alternative expressions for $\bar{S}_f$ available in HEC-2 are as follows:

Average Conveyance Equation

$$\bar{S}_f = \left(\frac{Q_1 + Q_2}{K_1 + K_2} \right)^2 \tag{6}$$

Average Friction Slope Equation

$$\bar{S}_f = \frac{S_{f_1} + S_{f_2}}{2} \tag{7}$$

Geometric Mean Friction Slope Equation

$$\bar{S}_f = \sqrt{S_{f_1} \cdot S_{f_2}} \tag{8}$$

Harmonic Mean Friction Slope Equation

$$\bar{S}_f = \frac{2\, S_{f_1} \cdot S_{f_2}}{S_{f_1} + S_{f_2}} \tag{9}$$

Equation (6), which has been used in HEC-2 since 1971, is the 'default' equation used by the program; that is, it is used automatically unless use of a different equation is requested by input. The program also contains an option by which one of the above equations is automatically selected on a reach by reach basis depending on flow regime and profile type (e.g., S1, M1, etc.). Further discussion of the alternative methods for evaluating friction loss is contained in Chapter 4, Optional Capabilities.

5. COMPUTATION PROCEDURE

The unknown water surface elevation at a cross section is determined by an iterative solution of equations (1) and (2). The computational procedure is as follows:

1. Assume a water surface elevation at the upstream cross section (or downstream cross section if a supercritical profile is being calculated).

2. Based on the assumed water surface elevation, determine the corresponding total conveyance and velocity head.

3. With values from step 2, compute $\bar{S}_f$ and solve equation (2) for h_e.

4. With values from steps 2 and 3, solve equation (1) for WS_2.

5. Compare the computed value of WS_2 with the values assumed in step 1; repeat steps 1 through 5 until the values agree to within .01 feet (or .01 meters).

Criteria used to assume water surface elevations in the iterative procedure varies from trial to trial. Generally the first trial is based on projecting the previous cross section's water surface elevation on the average of the friction slopes from the previous two cross sections. The second trial is an arithmetic average of the computed and assumed elevations from the first trial. The third and subsequent trials are generally based on a "secant" method of projecting the rate of change of the difference between computed and assumed elevations for the previous two trials to zero. The change from one trial to the next is constrained to a maximum of ±50% of the assumed depth from the previous trial.

Once a 'balanced' water surface elevation has been obtained for a cross section, checks are made to ascertain that the elevation is on the 'right' side of the critical water surface elevation (e.g., above the critical elevation if a subcritical profile is being calculated). If the balanced elevation is on the 'wrong' side of the critical water surface elevation, critical depth is assumed for the cross section and a message to that effect is printed by the program. The program user should be aware of critical depth assumptions and determine the reasons for their occurrence, because in many cases they result from reach lengths being too long or from misrepresentation of the effective flow areas of cross sections.

For a subcritical profile, a preliminary check for proper flow regime involves the following equation:

$$\left(\alpha \frac{V^2}{2g}\right)_{\text{test}} = \frac{A_t}{2T} \tag{10}$$

where:

$\left(\alpha \frac{V^2}{2g}\right)_{\text{test}}$ = velocity head that would exist if critical conditions existed at the balanced water surface elevation.

A_t = total flow area

T = water surface width

If the calculated velocity head, $\alpha \frac{V^2}{2g}$, is less than 94% of $\left(\alpha \frac{V^2}{2g}\right)_{\text{test}}$ the balanced water surface elevation will be accepted for the cross section. If the calculated velocity head is greater than 94% of the test value, the critical water surface elevation will be determined (by a procedure discussed in section 6) so that a direct comparison of balanced elevation versus critical elevation can be made.

For a supercritical profile, critical depth is automatically calculated for every cross section, which enables a direct comparison between balanced and critical elevations.

6. CRITICAL DEPTH DETERMINATION

Critical depth for a cross section will be determined if any of the following conditions are satisfied:

(1) The supercritical flow regime has been specified.

(2) Calculation of critical depth has been requested with the program's critical depth option.

(3) This is the first cross section and critical depth starting conditions have been specified.

(4) The critical depth check for a subcritical profile indicates that critical depth needs to be determined to verify the flow regime associated with the balanced elevation.

The total energy head for a cross section is defined by:

$$H = WS + \frac{\alpha V^2}{2g} \tag{11}$$

where:

H = total energy head

WS = water surface elevation

$\frac{\alpha V^2}{2g}$ = velocity head

The critical water surface elevation is the elevation for which the total energy head is a minimum. The critical elevation is determined with an iterative procedure whereby values of WS are assumed and corresponding values of H are determined with equation (11) until a minimum value for H is reached.

To speed the iteration process, a parabolic interpolation procedure is followed. The procedure basically involves determining values of H for three values of WS that are spaced at equal ΔWS intervals. The WS corresponding to the minimum value for H defined by a parabola passing through the three points (on the H versus WS plane) is used as the basis for the next assumption of a value for WS.

It is presumed that critical depth has been obtained when there is less than a 2.5% change in depth from one iteration to the next and provided the energy head has either decreased or has not increased by more than .01 feet. The tolerance of 2.5% may be changed by program input.

7. PROGRAM LIMITATIONS

The following assumptions are implicit in the analytical expressions used in the program:

(1) Flow is steady

(2) Flow is gradually varied

(3) Flow is one dimensional (i.e., velocity components in directions other than the direction of flow are not accounted for)

(4) River channels have 'small' slopes, say less than 1:10

Flow is assumed to be steady because time-dependent terms are not included in the energy equation (1). Flow is assumed to be gradually varied because equation (1) is based on the premise that a hydrostatic pressure distribution exists at each cross section. Flow is assumed to be one-dimensional because equation (4) is based on the premise that the total energy head is the same for all points in a cross section. Small channel slopes are assumed because the pressure head which is a component of WS in equation (1) is represented by the water depth measured vertically.

The program does not have the capability to deal with movable boundaries (i.e., sediment transport) and requires that energy losses be definable with the terms contained in equation (2) or by using criteria described in Appendix IV for bridge, culvert or weir flow.

Chapter 3

BASIC DATA REQUIREMENTS

1. GENERAL

A major portion of the programming in HEC-2 is devoted to providing a large variety of input and data manipulation options. The program objective is quite simple -- compute water surface elevations at all locations of interest for given flow values. The data needed to perform these computations include: flow regime, starting elevation, discharge, loss coefficients, cross section geometry, and reach lengths. The options available for providing and manipulating input are discussed in the following sections.

2. FLOW REGIME

Profile computations begin at a cross section with known or assumed starting conditions and proceed upstream for subcritical flow or downstream for supercritical flow. The direction of flow is specified on the J1 card (first job card) by setting variable IDIR equal to one for supercritical flow or zero (or blank) for subcritical flow. Subcritical profiles computed by the program (IDIR = 0) are constrained to critical depth or above, and supercritical profiles (IDIR = 1) are constrained to critical depth or below. The program will not allow profile computations to cross critical depth except for certain bridge analysis problems. In cases where flow passes from one flow regime to another as shown in Figure 2 below, it is necessary to compute the profile twice, alternately assuming subcritical and supercritical flow. Results of a subcritical profile, (shown as ⊙ in Figure 2) computed for the example stream would plot at critical depth (above the actual water surface profile) in the steep reach of stream. Results from a supercritical profile computation (shown as x in Figure 2)

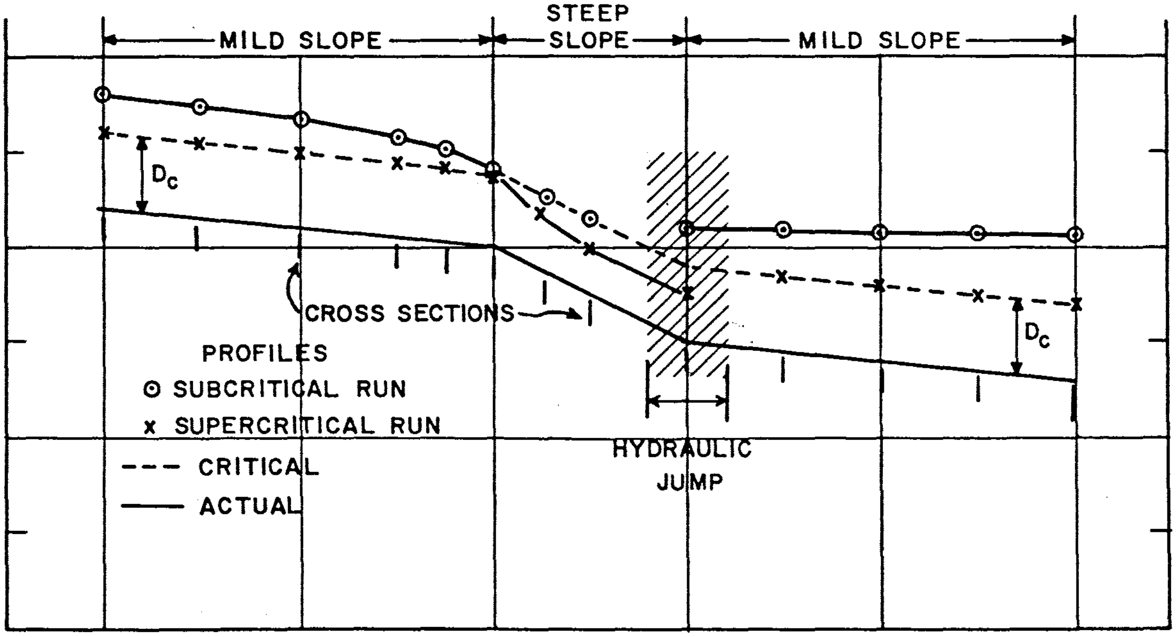

Figure 2. Profiles Calculated for Subcritical and Supercritical Flows

would plot at critical depth (below the actual water surface profile) for both mild reaches of the stream. The final plotted profile should incorporate computed results from both computations and an analysis of the hydraulic jump. HEC-2 does not contain the capability to determine the position of the hydraulic jump or energy losses associated with the jump.

3. STARTING ELEVATION

The water surface elevation for the beginning cross section may be specified in one of three ways: (1) as critical depth, (2) as a known elevation, (3) by the slope area method. By setting the variable STRT on the J1 card equal to minus one, critical depth will be computed and used as the starting water surface elevation. This method is appropriate at locations where critical or near critical conditions are known to exist for the range of discharges being computed, e.g., a waterfall, weir or a section of rapids. When a rating curve is available, the appropriate starting elevation can be specified by variable WSEL on the J1 card.

For beginning by the slope area method, STRT is set equal to the estimated slope of the energy grade line (must be a positive value) and WSEL is used as the initial estimate of the water surface elevation. The flows computed for the fixed slope are compared with the starting flow and the depth is adjusted until the computed flow is within one percent of the starting flow. The water surface elevation thus determined is used as the starting water surface elevation for subsequent water surface profile computations.

4. DISCHARGE

Discharge may be specified and altered in several ways. The Variable Q on the first job card (J1 card) specifies the starting discharge for single profile runs. When it is desired to change the discharge for a single profile run, the variable QNEW on the X2 card can be used to permanently change the discharge at any cross section.

An alternate procedure utilizes the QT cards (discharge table) and may be used to specify from one to nineteen discharge values for single or multiple profile runs. QT cards may be used to specify starting discharges and to permanently change discharges at any cross section in a data set. Variable INQ on the J1 card directs the program to the field of the QT card that contains the discharge for that profile. When a value of FQ is entered, all discharges on the X2 cards and discharges in the specified INQ of the QT cards are multiplied by the value.

5. ENERGY LOSS COEFFICIENTS

Several types of loss coefficients are utilized by the program to evaluate head losses: (1) Manning's 'n' values for friction loss, (2) contraction and expansion coefficients to evaluate transition (shock) losses, and (3) bridge loss coefficients to evaluate losses related to weir shape, pier configuration, and pressure flow.

Because Manning's coefficient of roughness 'n' depends on such factors as type and amount of vegetation, channel configuration and stage, several options are available to vary 'n'. When three 'n' values are sufficient to describe the channel and overbank roughness, the first three fields of the NC card ('n' value change) are used. Any of the 'n' values may be permanently changed at any cross section by using another NC card. Often three values are not enough to adequately describe the lateral roughness variation in the cross section; in this case the NH card ('n' value - horizontal) is used. The number of 'n' values used to describe the roughness is entered as variable NUMNH in the first field and the 'n' values and corresponding cross section stations are entered in subsequent fields. These 'n' values will be used for all subsequent cross sections unless changed by another NH card. Normally the NH card 'n' values should be redefined for each cross section with new geometry.

Data indicating the variation of Manning's 'n' with river stage may be used in the program. Manning's 'n' and the corresponding stage elevation (beginning with the lowest elevation) are entered on the NV card ('n' value - vertical), beginning in the second and third fields, respectively. Variable NUMNV in field one is the number of 'n' values input on the NV cards. This 'n' value option applies only to the channel area.

If for subsequent jobs of the same run it is desired to modify the 'n' values specified on the NC, NH, and NV cards by a factor, variable FN on the J2 card may be used. The desired factor is entered as variable FN for each job. If the value of FN is negative, the factor is multiplied by the channel 'n' values on the NC card but the overbank 'n' values are not changed.

Reference 1 shows Manning's 'n' values for average channels. A more extensive compilation of 'n' values for streams and flood plains is listed in reference 2. References 3 and 4 use pictures of selected streams as a guide to 'n' value determination.

Contraction or expansion of flow due to changes in the channel cross section is a common cause of energy losses within a reach. Whenever this occurs, the loss may be computed by specifying the contraction and expansion coefficients as variables CCHV and CEHV, respectively, on the NC card. The coefficients are multiplied by the absolute difference in velocity heads between the cross sections to give the energy loss caused by the transition. Where the change in river cross section is small, coefficients CCHV and CEHV are typically on the order of 0.1 and 0.3, respectively. When the change in effective cross section area is abrupt such as at bridges, CCHV and CEHV may be as high as 0.6 and 1.0, respectively. These values may be changed at any cross section by inserting a new NC card. These new values will be used until changed again by another NC card. For additional information concerning transition losses and for information on bridge loss coefficients see Appendix IV.

6. CROSS SECTION GEOMETRY

Boundary geometry for the analysis of flow in natural streams is specified in terms of ground surface profiles (cross sections) and the measured distances between them (reach lengths). Cross sections are located at intervals along a stream to characterize the flow carrying capability of the stream and its adjacent flood plains. They should extend across the entire flood plain and should be perpendicular to the anticipated flow lines (approximately perpendicular to contour lines). Occasionally it is necessary to layout cross sections in a curved or dog-leg alignment ot meet this requirement. Every effort should be made to obtain cross sections that accurately represent the stream and flood plain geometry. However, ineffective flow areas of the flood plain such as stream inlets, small ponds or indents in the valley floor should generally not be included in the cross section geometry.

Cross sections are required at representative locations throughout a stream reach and at locations where changes occur in discharge, slope, shape, or roughness, at locations where levees begin or end and at bridges or control structures such as weirs. Where abrupt changes occur, several cross sections should be used to describe the change regardless of the distance. Cross section spacing is also a function of stream size, slope, and the uniformity of cross section shape. In general, large uniform rivers of flat slope normally require the fewest number of cross sections per mile. The purpose of the study also affects spacing of cross sections. For instance, navigation studies on large relatively flat streams may require closely spaced (e.g., five hundred feet) cross sections to analyze the effect of local conditions on low flow depths, whereas cross sections for sedimentation studies to determine deposition in reservoirs may be spaced at intervals of up to five or ten miles.

The choice of friction loss equation may also influence the spacing of cross sections. For instance, cross section spacing may be maximized when calculating an M1 profile with the average friction slope equation or when the harmonic mean friction slope equation is used to compute M2 profiles.

Each cross section in an HEC-2 data set is identified and described by X1 and GR cards. Variable SECNO on the X1 card is the cross section identification number which may correspond to stationing along the channel, mile points, or any fictitious numbering system, since it is only used to identify output and is not used in the computations. Each data point in the cross section is given a station number corresponding to the horizontal distance from a zero point on the left. The elevation and a corresponding station number of each data point are input as variables EL(I) and STA(I) on GR cards. Up to one hundred data points may be used to describe cross section geometry for most program applications. When the encroachment options are utilized, no more than ninety-five data points should be used, since they generate additional data points automatically to define the encroachment limits. The channel improvement option also should be used with few than one hundred data points since it will generate data points (four or more depending on the geometry).

Cross section data is traditionally oriented looking downstream since the program considers the left side of the stream to have the lowest station numbers and the right side to have the highest. The left and right stations separating the channel from the overbank areas are specified as variables STCHL and STCHR on the X1 card. End points of a cross section that are too low (below the computed water surface elevation) will automatically be extended vertically and a note indicating the extension amount will be printed.

Numerous program options are available to allow the user to easily add or modify cross section data. For example, when the user wishes to repeat a surveyed cross section, a single X1 card may be input to identify the cross section and to provide reach length information. X1 card variables, PXSECR and PXSECE, allow the user to modify the horizontal and vertical dimensions of the repeated cross section data. Other program options to modify cross section data to model improved channel sections, encroachments and ineffective flow areas are described in detail in the following chapter.

7. REACH LENGTHS

The measured distances between cross sections are referred to as reach lengths. The reach lengths for the left overbank, right overbank and channel used in computations are specified on the X1 card by variables XLOBL, XLOBR, and XLCH, respectively. Channel reach lengths are typically measured along the thalweg. Overbank reach lengths should be measured along the anticipated path of the center of mass of the overbank flow. Often these three values will be equal. There are, however, conditions where they will differ, such as at river bends, or where the channel meanders considerably and the overbanks are straight. Where the distances between cross sections for channel and overbanks are different, a discharge-weighted reach length is determined based on the discharges in the main channel and left and right overbank segments of the reach (see Equation 3, page 4).

CHAPTER 1

OPEN-CHANNEL FLOW AND ITS CLASSIFICATIONS

1-1. Description. The flow of water in a conduit may be either *open-channel flow* or *pipe flow*. The two kinds of flow are similar in many ways but differ in one important respect. Open-channel flow must have a *free surface*, whereas pipe flow has none, since the water must fill the whole conduit. A free surface is subject to atmospheric pressure. Pipe flow, being confined in a closed conduit, exerts no direct atmospheric pressure but hydraulic pressure only.

The two kinds of flow are compared in Fig. 1-1. Shown on the left side is pipe flow. Two piezometer tubes are installed on the pipe at sections 1 and 2. The water levels in the tubes are maintained by the pressure in the pipe at elevations represented by the so-called ***hydraulic grade line***. The pressure exerted by the water in each section of the pipe is indicated in the corresponding tube by the height y of the water column above the center line of the pipe. The total energy in the flow of the section with reference to a datum line is the sum of the elevation z of the pipe-center line, the piezometric height y, and the velocity head $V^2/2g$, where V is the mean velocity of flow.[1] The energy is represented in the figure by what is called the *energy grade line* or simply the *energy line*. The loss of energy that results when water flows from section 1 to section 2 is represented by h_f. A similar diagram for open-channel flow is shown on the right side of Fig. 1-1. For simplicity, it is assumed that the flow is parallel and has a uniform velocity distribution and that the slope of the channel is small. In this case, the water surface is the hydraulic grade line, and the depth of the water corresponds to the piezometric height.[2]

Despite the similarity between the two kinds of flow, it is much more difficult to solve problems of flow in open channels than in pressure pipes. Flow conditions in open channels are complicated by the fact that the position of the free surface is likely to change with respect to time and space and also by the fact that the depth of flow, the discharge, and the slopes of the channel bottom and of the free surface are interdependent. Reliable experimental data on flow in open channels are usually difficult to obtain. Furthermore, the physical condition of open channels varies much more widely than that of pipes. In pipes the cross section of flow is fixed, since it is completely defined by the geometry of the conduit. The cross section of a pipe is generally round, but that of an open channel may be of any shape—from the circular to the irregular forms of natural streams. In pipes, the interior surface ordinarily ranges in roughness from that of new smooth brass or wooden-stave pipes, on the one hand, to that of old corroded iron or steel pipes, on the other. In open channels the surface varies from that of the polished metal used in testing flumes to that of rough irregular river beds. Moreover, the roughness in an open channel varies with the position of the free surface. Therefore, the selection of friction coefficients is attended by greater uncertainty for open channels than for pipes. In general, the treatment of open-channel flow is somewhat more empirical than that of pipe flow. The empirical method is the best available at present and, if cautiously applied, can yield results of practical value.

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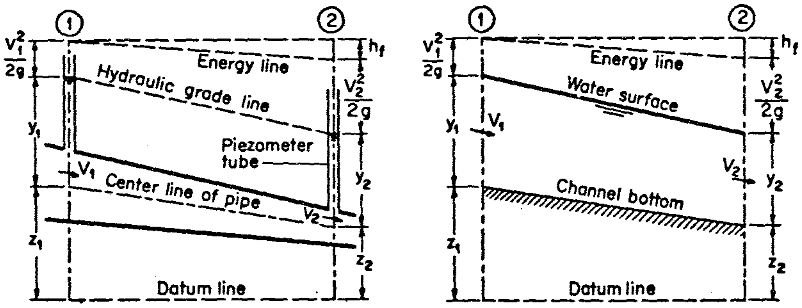
[1] It is here assumed that the velocity is uniformly distributed across the conduit section; otherwise a correction would have to be made, such as is described in Art. 2-7 for open channels.

[2] If the flow were curvilinear or if the slope of the channel were large, the piezometric height would be appreciably different from the depth of flow (Arts. 2-9 and 2-10). As a result, the hydraulic grade line would not coincide exactly with the water surface.



Pipe flow

Open-channel flow

FIG. 1-1. Comparison between pipe flow and open-channel flow.

The flow in a closed conduit is not necessarily pipe flow. It must be classified as open-channel flow if it has a free surface. The storm sewer, for example, which is a closed conduit, is generally designed for open-channel flow because the flow in the sewer is expected to maintain a free surface most of the time.

1-2. Types of Flow. Open-channel flow can be classified into many types and described in various ways. The following classification is made according to the change in flow depth with respect to time and space.

Steady Flow and Unsteady Flow: Time as the Criterion. Flow in an open channel is said to be *steady* if the depth of flow does not change or if it can be assumed to be constant during the time interval under consideration. The flow is *unsteady* if the depth changes with time. In most open-channel problems it is necessary to study flow behavior only under steady conditions. If, however, the change in flow condition with respect to time is of major concern, the flow should be treated as unsteady. In floods and surges, for instance, which are typical examples of unsteady flow, the stage of flow changes instantaneously as the waves pass by, and the time element becomes vitally important in the design of control structures.

For any flow, the discharge Q at a channel section is expressed by

$$Q = VA \tag{1-1}$$

where V is the mean velocity and A is the flow cross-sectional area normal to the direction of the flow, since the mean velocity is defined as the discharge divided by the cross-sectional area.

In most problems of steady flow the discharge is constant throughout the reach of the channel under consideration; in other words, the flow is *continuous*. Thus, using Eq. (1-1),

$$Q = V_1A_1 = V_2A_2 = \cdots \tag{1-2}$$

where the subscripts designate different channel sections. This is the *continuity equation* for a continuous steady flow.

Equation (1-2) is obviously invalid, however, where the discharge of a steady flow is *nonuniform* along the channel, that is, where water runs in or out along the course of flow. This type of flow, known as *spatially varied or discontinuous flow,* is found in roadside gutters, side-channel spillways, the washwater troughs in filters, the effluent channels around sewage-treatment tanks, and the main drainage channels and feeding channels in irrigation systems.

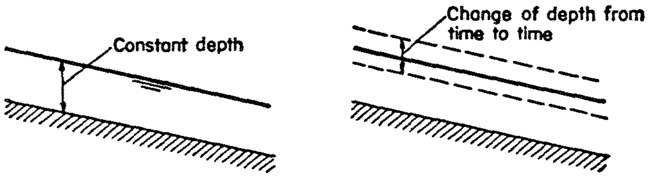
The law of continuity of unsteady flow requires consideration of the time effect. Hence, the continuity equation for continuous unsteady flow should include the time element as a variable (Art. 18-1).

Uniform Flow and Varied Flow: Space as the Criterion. Open-channel flow is said to be *uniform* if the depth of flow is the same at every section of the channel. A uniform flow may be steady or unsteady, depending on whether or not the depth changes with time.

Steady uniform flow is the fundamental type of flow treated in open-channel hydraulics. The depth of the flow does not change during the time interval under consideration. The establishment of *unsteady uniform flow* would require that the water surface fluctuate from time to time while remaining parallel to the channel bottom. Obviously, this is a practically impossible condition. The term "uniform flow" is, therefore, used hereafter to refer only to steady uniform flow.

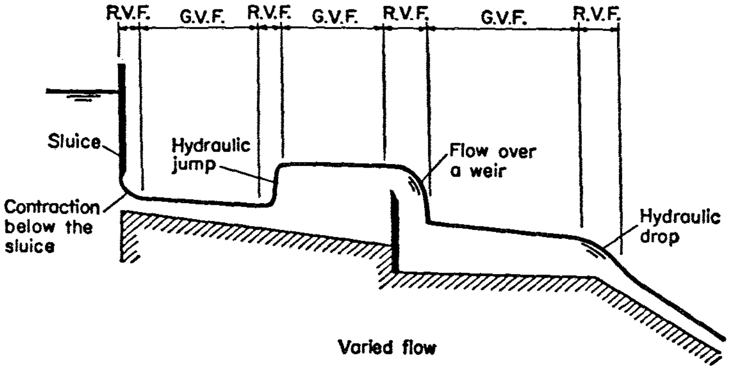
Flow is *varied* if the depth of flow changes along the length of the channel. Varied flow may be either steady or unsteady. Since unsteady uniform flow is rare, the term "unsteady flow" is used hereafter to designate *unsteady varied flow* exclusively.

Varied flow may be further classified as either *rapidly* or *gradually varied*. The flow is rapidly varied if the depth changes abruptly over a comparatively short distance; otherwise, it is gradually varied. A rapidly varied flow is also known as a *local phenomenon;* examples are the hydraulic jump and the hydraulic drop.

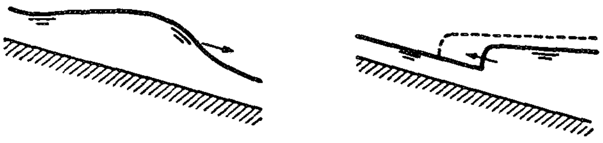


Uniform flow – Flow in a laboratory channel

Unsteady uniform flow – Rare



Varied flow



G.V.F. – Flood wave

R.V.F. – Bore

Unsteady flow

Fig. 1-2. Various types of open-channel flow. G.V.F. = gradually varied flow; R.V.F. = rapidly varied flow.

For clarity, the classification of open-channel flow is summarized as follows:

A. Steady flow
  1. Uniform flow
  2. Varied flow
    a. Gradually varied flow
    b. Rapidly varied flow
B. Unsteady flow
  1. Unsteady uniform flow (rare)
  2. Unsteady flow (i.e., unsteady varied flow)
    a. Gradually varied unsteady flow
    b. Rapidly varied unsteady flow

Various types of flow are sketched in Fig. 1-2. For illustrative purposes, these diagrams, as well as other similar sketches of open channels in this book, have been drawn to a greatly exaggerated vertical scale, since ordinary channels have small bottom slopes.

1-3. State of Flow. The state or behavior of open-channel flow is governed basically by the effects of viscosity and gravity relative to the inertial forces of the flow. The surface tension of water may affect the behavior of flow under certain circumstances, but it does not play a significant role in most open-channel problems encountered in engineering.

Effect of Viscosity. Depending on the effect of viscosity relative to inertia, the flow may be laminar, turbulent, or transitional.

The flow is *laminar* if the viscous forces are so strong relative to the inertial forces that viscosity plays a significant part in determining flow behavior. In laminar flow, the water particles appear to move in definite smooth paths, or streamlines, and infinitesimally thin layers of fluid seem to slide over adjacent layers.

The flow is *turbulent* if the viscous forces are weak relative to the inertial forces. In turbulent flow, the water particles move in irregular paths which are neither smooth nor fixed but which in the aggregate still represent the forward motion of the entire stream.

Between the laminar and turbulent states there is a mixed, or *transitional*, state.

The effect of viscosity relative to inertia can be represented by the *Reynolds number*, defined as

$$\mathrm{R} = \frac{VL}{\nu} \tag{1-3}$$

where V is the velocity of flow in fps; L is a characteristic length in ft, here considered equal to the hydraulic radius R of a conduit; and ν (nu) is the kinematic viscosity of water in ft^2/sec. The kinematic viscosity

As the flow in most channels is turbulent, a model employed to simulate a prototype channel should be designed so that the Reynolds number of flow of the model channel is in the turbulent range.

Effect of Gravity. The effect of gravity upon the state of flow is represented by a ratio of inertial forces to gravity forces. This ratio is given by the *Froude number*,[1] defined as

$$\mathbf{F} = \frac{V}{\sqrt{gL}} \tag{1-10}$$

where V is the mean velocity of flow in fps, g is the acceleration of gravity in ft/sec^2, and L is a characteristic length in ft. In open-channel flow the characteristic length is made equal to the *hydraulic depth* D, which is defined as the cross-sectional area of the water normal to the direction of flow in the channel divided by the width of the free surface. For rectangular channels this is equal to the depth of the flow section.

When $\mathbf{F}$ is equal to unity, Eq. (1-10) gives

$$V = \sqrt{gD} \tag{1-11}$$

and the flow is said to be in a *critical* state. If $\mathbf{F}$ is less than unity, or $V < \sqrt{gD}$, the flow is *subcritical*. In this state the role played by gravity forces is more pronounced; so the flow has a low velocity and is often described as tranquil and streaming. If $\mathbf{F}$ is greater than unity, or $V > \sqrt{gD}$, the flow is *supercritical*. In this state the inertial forces become dominant; so the flow has a high velocity and is usually described as rapid, shooting, and torrential.

In the mechanics of water waves, the critical velocity $\sqrt{gD}$ is identified as the *celerity* of the small gravity waves that occur in shallow water in channels as a result of any momentary change in the local depth of the water (Art. 18-6). Such a change may be developed by disturbances or obstacles in the channel that cause a displacement of water above and below the mean surface level and thus create waves that exert a weight or gravity force. It should be noted that a gravity wave can be propagated upstream in water of subcritical flow but not in water of supercritical flow, since the celerity is greater than the velocity of flow in the former case and less in the latter. Therefore, the possibility or impossibility of propagating a gravity wave upstream can be used as a criterion for distinguishing between subcritical and supercritical flow.

Since the flow in most channels is controlled by the gravity effect, a model used to simulate a prototype channel for testing purposes must be designed for this effect; that is, the Froude number of the flow in the model channel must be made equal to that of the flow in the prototype channel.

---

[1] Other dimensionless ratios used for the same purpose include (1) the *kinetic-flow factor* $\lambda = V^2/gL = \mathbf{F}^2$, first used by Rehbock [25] and then by Bakhmeteff [26]; (2) the *Boussinesq number* $\mathrm{B} = V/\sqrt{2gR}$, first used by Engel [27]; and (3) the *kineticity* or *velocity-head ratio* $k = V^2/2gL$, proposed by Stevens [28] and Posey [29], respectively.

1-4. Regimes of Flow. A combined effect of viscosity and gravity may produce any one of four *regimes of flow* in an open channel, namely, (1) *subcritical-laminar*, when F is less than unity and R is in the laminar range; (2) *supercritical-laminar*, when F is greater than unity and R is in the laminar range; (3) *supercritical-turbulent*, when F is greater than unity and R is in the turbulent range; and (4) *subcritical-turbulent*, when F is less than unity and R is in the turbulent range. The depth-velocity relationships for the four flow regimes in a wide open channel can be shown by a logarithmic plot (Fig. 1-5) [30]. The heavy line for F = 1 and the shaded band for the laminar-turbulent transitional range intersect on the graph and divide the whole area into four portions, each of which represents a flow regime. The first two regimes, subcritical-laminar and supercritical-laminar, are not commonly encountered in applied open-channel hydraulics, since the flow is generally turbulent in the channels considered in engineering problems. However, these regimes occur frequently where there is very thin depth—this is known as *sheet flow*—and they become significant in such problems as the testing of hydraulic models, the study of overland flow, and erosion control for such flow.

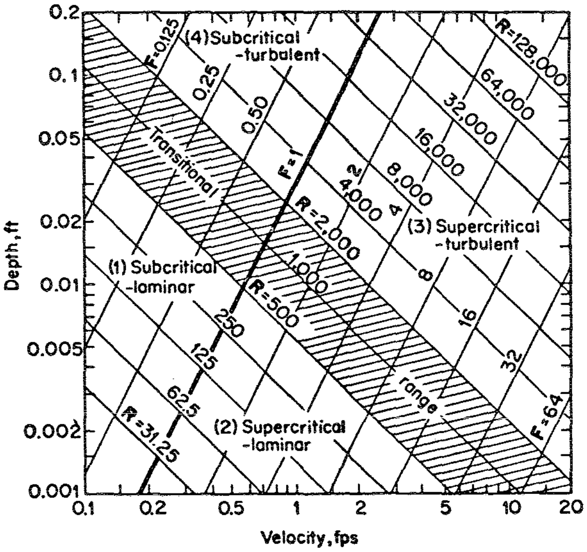


FIG. 1-5. Depth-velocity relationships for four regimes of open-channel flow. (*After Robertson and Rouse* [30].)

Photographs of the four regimes of flow are shown in Fig. 1-6. In each

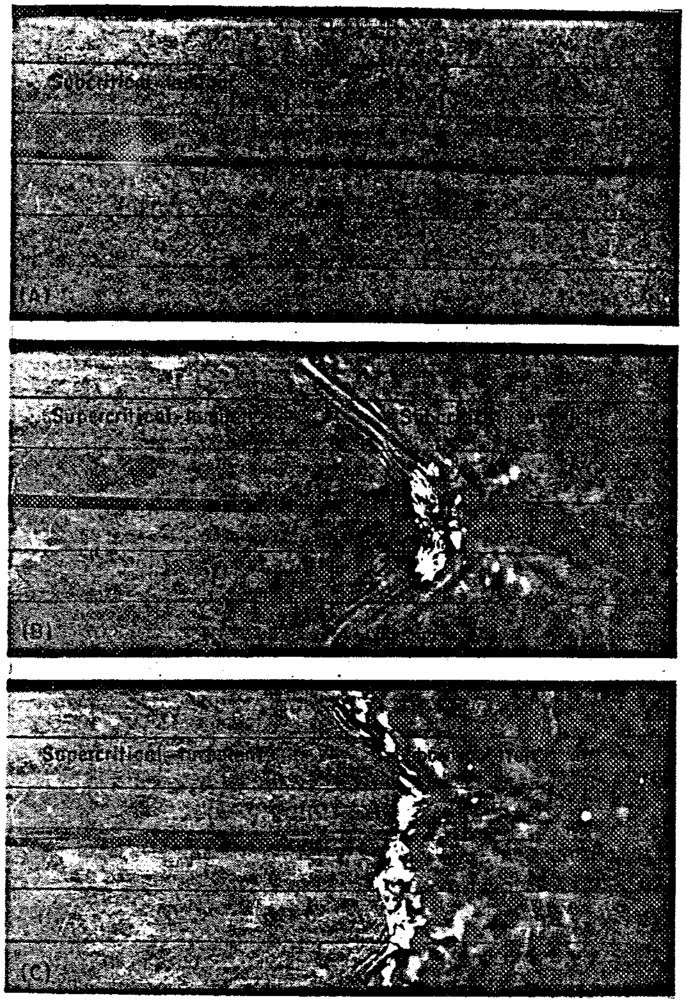


FIG. 1-6. Photographs showing four flow regimes in a laboratory channel. (*Courtesy of H. Rouse.*)

photograph the direction of flow is from left to right. All flows are uniform except those on the right side of the middle and bottom views. The top view represents uniform subcritical-laminar flow. The flow is subcritical, since the Froude number was adjusted to slightly below the critical value; and the streak of undiffused dye indicates that it is laminar. The middle view shows a uniform supercritical-laminar flow changing to varied subcritical-turbulent. The bottom view shows a uniform supercritical-turbulent flow changing to varied subcritical-turbulent. In both cases, the diffusion of dye is the evidence of turbulence.

It is believed that gravity action may have a definitive effect upon the flow resistance in channels at the turbulent-flow range. The experimental data studied by Jegorow [31] and Iwagaki [32] for smooth rectangular channels and by Hom-ma [33] for rough channels have shown that, in the supercritical-turbulent regime of flow, the friction factor is likely to increase with increasing Froude number. Generally, the effect of gravity on the value of f is practically negligible where the Froude number is small, say, less than 3. A further investigation by Iwagaki [34] indicates that, with increasing Froude number, the friction factor of turbulent flow in both smooth and rough open channels becomes larger than that in pipes. It is possible that the presence of the free surface in open-channel flow makes the channel hydraulically rougher than the pipe. When more data and evidence become available, the Froude number, representing the gravity effect, may have to be considered as an additional factor in defining the f-R relationship for supercritical-turbulent flow.

PROBLEMS

1-1. With reference to Fig. 1-1, show that the theoretical discharge of the open-channel flow may be expressed by

$$Q = A_2 \sqrt{\frac{2g(\Delta y - h_f)}{1 - (A_2/A_1)^2}} \qquad \text{(1-12)}$$

where A_1 and A_2 are the cross-sectional areas of the flow at sections 1 and 2, respectively, and Δy is the drop in water surface between the sections.

1-2. Verify Eq. (1-10).

1-3. Verify by computation the depth-velocity relationships shown in Fig. 1-5 for the four flow regimes in a wide rectangular open channel. The temperature of the water is taken as 68°F.

1-4. A model channel is used to simulate a prototype channel 100 ft wide, carrying a discharge of 500 cfs at a depth of 4 ft. The model is designed for gravity effect, and a turbulent-flow condition is assured. Determine the minimum size of the model and the scale ratio, assuming the upper limit of the transitional-flow region to be $\mathbf{R} = 2{,}000$. The scale ratio is the ratio of the linear dimension of the model to that of the prototype.

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CHAPTER 2

OPEN CHANNELS AND THEIR PROPERTIES

2-1. Kinds of Open Channel. An open channel is a conduit in which water flows with a free surface. Classified according to its origin a channel may be either *natural* or *artificial.*

Natural channels include all watercourses that exist naturally on the earth, varying in size from tiny hillside rivulets, through brooks, streams, small and large rivers, to tidal estuaries. Underground streams carrying water with a free surface are also considered natural open channels.

The hydraulic properties of natural channels are generally very irregular. In some cases empirical assumptions reasonably consistent with actual observations and experience may be made such that the conditions of flow in these channels become amenable to the analytical treatment of theoretical hydraulics. A comprehensive study of the behavior of flow in natural channels requires knowledge of other fields, such as hydrology, geomorphology, sediment transportation, etc. It constitutes, in fact, a subject of its own, known as *river hydraulics.*

Artificial channels are those constructed or developed by human effort: navigation channels, power canals, irrigation canals and flumes, drainage ditches, trough spillways, floodways, log chutes, roadside gutters, etc., as well as model channels that are built in the laboratory for testing purposes. The hydraulic properties of such channels can be either controlled to the extent desired or designed to meet given requirements. The application of hydraulic theories to artificial channels will, therefore, produce results fairly close to actual conditions and, hence, are reasonably accurate for practical design purposes.

Under various circumstances in engineering practice the artificial open channel is given different names, such as "canal," "flume," "chute," "drop," "culvert," "open-flow tunnel," etc. These names, however, are used rather loosely and can be defined only in a very general way. The *canal* is usually a long and mild-sloped channel built in the ground, which may be unlined or lined with stone masonry, concrete, cement, wood, or bituminous materials. The *flume* is a channel of wood, metal, concrete, or masonry, usually supported on or above the surface of the ground to carry water across a depression. The *chute* is a channel having steep slopes. The *drop* is similar to a chute, but the change in elevation is effected in a short distance. The *culvert* flowing partly full is a covered channel of comparatively short length installed to drain water through highway and railroad embankments. The *open-flow tunnel* is a comparatively long covered channel used to carry water through a hill or any obstruction on the ground.

2-2. Channel Geometry. A channel built with unvarying cross section and constant bottom slope is called a *prismatic channel.* Otherwise, the channel is *nonprismatic;* an example is a trough spillway having variable width and curved alignment. Unless specifically indicated, the channels described in this book are prismatic.

The term *channel section* used in this book refers to the cross section of a channel taken normal to the direction of the flow. A *vertical channel section,* however, is the vertical section passing through the lowest or bottom point of the channel section. For horizontal channels, therefore, the channel section is always a vertical channel section.

Natural channel sections are in general very irregular, usually varying from an approximate parabola to an approximate trapezoid. For streams subject to frequent floods, the channel may consist of a main channel section carrying normal discharges and one or more side channel sections for accommodating overflows.

Artificial channels are usually designed with sections of regular geometric shapes. Table 2-1 lists seven geometric shapes that are in common use. The trapezoid is the commonest shape for channels with unlined earth banks, for it provides side slopes for stability. The rectangle and triangle are special cases of the trapezoid. Since the rectangle has vertical sides, it is commonly used for channels built of stable materials, such as lined masonry, rocks, metal, or timber. The triangular section is used only for small ditches, roadside gutters, and laboratory works. The circle is the popular section for sewers and culverts of small and medium sizes. The parabola[1] is used as an approximation of sections of small and medium-size natural channels. The round-cornered rectangle is a modification of the rectangle. The round-bottom triangle is an approximation of the parabola; it is a form usually created by excavation with shovels.

Closed geometric sections other than the circle are frequently used in sewerage, particularly for sewers large enough for a man to enter. These sections are given various names according to their form; they may be egg-shaped, ovoid, semielliptical, *U*-shaped, catenary, horseshoe, basket-handle, etc. The complete rectangle and square are also common for large sewers. Dimensions and properties of sewer sections may be found in textbooks on sewerage.[1]

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[1] The side slope z:1 of a parabolic section at the intersection of the sides with the free surface can be computed easily by the simple formula $z = T/4y$.

Russian engineers [1] also use semielliptical and parabolic sections of higher order: $y = ax^p$ with $p = 3$ or 4. The constant a is computed from the side slope assumed at the free surface.

A special geometric section known as *hydrostatic catenary* or *lintearia* [4,5] is the shape of the cross section of a trough, formed of flexible sheets assumed to be weightless, filled with water up to the top of the section, and firmly supported at the upper edges of the sides but with no effects of fixation. The hydrostatic catenary has been used for the design of the sections of some elevated irrigation flumes. These flumes are constructed of metal plates so thin that their weight is negligible, and are firmly attached to beams at the upper edges.

2-3. Geometric Elements of Channel Section. *Geometric elements* are properties of a channel section that can be defined entirely by the geometry of the section and the depth of flow. These elements are very important and are used extensively in flow computations.

For simple regular channel sections, the geometric elements can be expressed mathematically in terms of the depth of flow and other dimensions of the section. For complicated sections and sections of natural streams, however, no simple formula can be written to express these elements, but curves representing the relation between these elements and the depth of flow can be prepared for use in hydraulic computations.

The definitions of several geometric elements of basic importance are given below. Other geometric elements used in this book will be defined where they first appear.

The *depth of flow* y is the vertical distance of the lowest point of a channel section from the free surface. This term is often used interchangeably with the *depth of flow section* d. Strictly speaking, the depth of flow section is the depth of flow normal to the direction of flow, or the height of the channel section containing the water. For a channel with a longitudinal slope angle θ, it can be seen that the depth of flow is equal to the depth of flow section divided by $\cos \theta$. In the case of steep channels, therefore, the two terms should be used discriminately.

The *stage* is the elevation or vertical distance of the free surface above a datum. If the lowest point of the channel section is chosen as the datum, the stage is identical with the depth of flow.

The *top width* T is the width of channel section at the free surface.

The *water area* A is the cross-sectional area of the flow normal to the direction of flow.

The *wetted perimeter* P is the length of the line of intersection of the channel wetted surface with a cross-sectional plane normal to the direction of flow.

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[1] Many typical sewer sections are described in [2] and [3].

The *hydraulic radius* R is the ratio of the water area to its wetted perimeter, or

$$R = \frac{A}{P} \tag{2-1}$$

The *hydraulic depth* D is the ratio of the water area to the top width, or

$$D = \frac{A}{T} \tag{2-2}$$

The *section factor for critical-flow computation* Z is the product of the water area and the square root of the hydraulic depth, or

$$Z = A\sqrt{D} = A\sqrt{\frac{A}{T}} \tag{2-3}$$

The *section factor for uniform-flow computation* $AR^{2/3}$ is the product of the water area and the two-thirds power of the hydraulic radius.

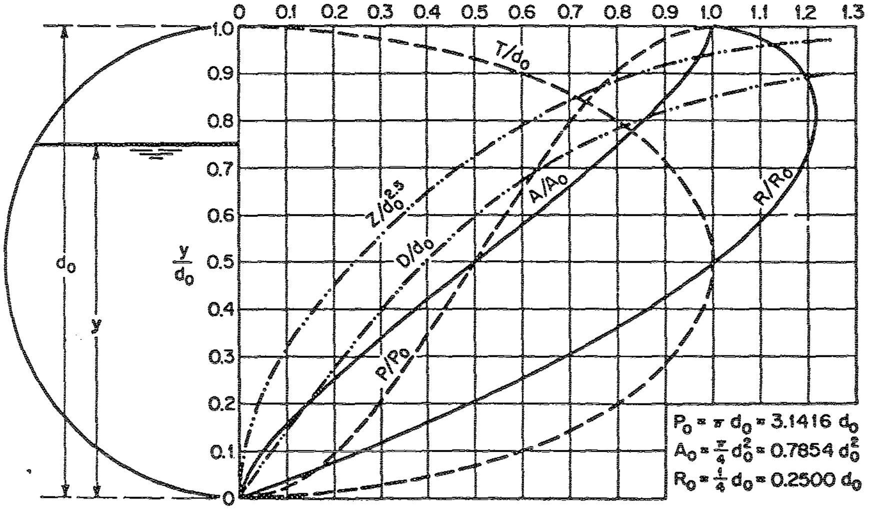

FIG. 2-1. Geometric elements of a circular section.

Table 2-1 furnishes a list of formulas for six basic geometric elements of seven commonly used channel sections. For a circular section, the curves in Fig. 2-1 represent the ratios of the geometric elements of the section to the corresponding elements when the section is flowing full. These curves are prepared from a table given in Appendix A. For certain trapezoidal, triangular, and parabolic sections commonly found in practical uses, the diagrams given in Appendix B provide a convenient means of determining the geometric elements.

CHAPTER 3

ENERGY AND MOMENTUM PRINCIPLES

3-1. Energy in Open-channel Flow. It is known in elementary hydraulics that the total energy in foot-pounds per pound of water in any streamline passing through a channel section may be expressed as the total head in feet of water, which is equal to the sum of the elevation above a datum, the pressure head, and the velocity head. For example, with respect to the datum plane, the total head H at a section O containing point A on a streamline of flow in a channel of large slope (Fig. 3-1) may be written

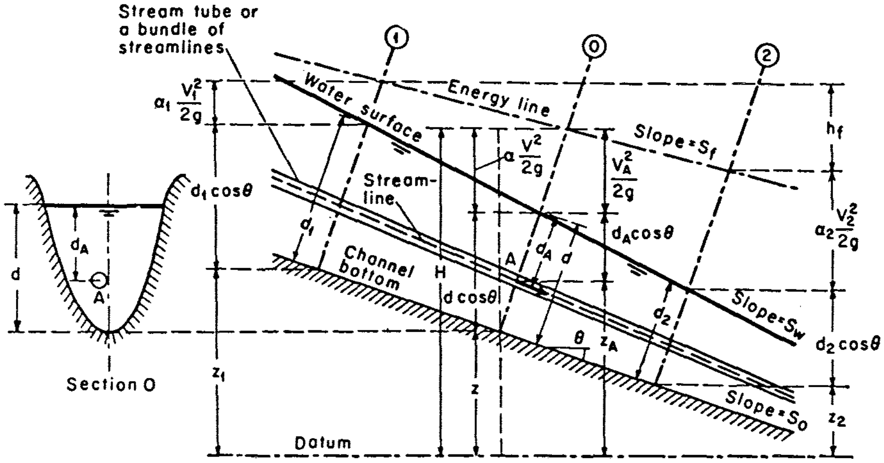

FIG. 3-1. Energy in gradually varied open-channel flow.

$$H = z_A + d_A \cos\theta + \frac{V_A^2}{2g} \tag{3-1}$$

where z_A is the elevation of point A above the datum plane, d_A is the depth of point A below the water surface measured along the channel section, θ is the slope angle of the channel bottom, and $V_A^2/2g$ is the velocity head of the flow in the streamline passing through A.

In general, every streamline passing through a channel section will have a different velocity head, owing to the nonuniform velocity distribution in actual flow. Only in an ideal parallel flow of uniform velocity distribution can the velocity head be truly identical for all points on the cross section. In the case of gradually varied flow, however, it may be assumed, for practical purposes, that the velocity heads for all points on the channel section are equal, and the energy coefficient may be used to correct for the over-all effect of the nonuniform velocity distribution. Thus, the total energy at the channel section is

$$H = z + d \cos \theta + \alpha \frac{V^2}{2g} \tag{3-2}$$

For channels of small slope, $\theta \approx 0$. Thus, the total energy at the channel section is

$$H = z + d + \alpha \frac{V^2}{2g} \tag{3-3}$$

Consider now a prismatic channel of large slope (Fig. 3-1). The line representing the elevation of the total head of flow is the energy line. The slope of the line is known as the *energy gradient*, denoted by S_f. The slope of the water surface is denoted by S_w and the slope of the channel bottom[1] by $S_0 = \sin \theta$. In uniform flow, $S_f = S_w = S_0 = \sin \theta$.

According to the principle of conservation of energy, the total energy head at the upstream section 1 should be equal to the total energy head at the downstream section 2 plus the loss of energy h_f between the two sections; or

$$z_1 + d_1 \cos \theta + \alpha_1 \frac{V_1^2}{2g} = z_2 + d_2 \cos \theta + \alpha_2 \frac{V_2^2}{2g} + h_f \tag{3-4}$$

This equation applies to parallel or gradually varied flow. For a channel of small slope, it becomes

$$z_1 + y_1 + \alpha_1 \frac{V_1^2}{2g} = z_2 + y_2 + \alpha_2 \frac{V_2^2}{2g} + h_f \tag{3-5}$$

Either of these two equations is known as the *energy equation*. When $\alpha_1 = \alpha_2 = 1$ and $h_f = 0$, Eq. (3-5) becomes

$$z_1 + y_1 + \frac{V_1^2}{2g} = z_2 + y_2 + \frac{V_2^2}{2g} = \text{const} \tag{3-6}$$

This is the well-known *Bernoulli energy equation*.[2]

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[1] The slope is generally defined as tan θ. For the present purpose, however, it is defined as sin θ.

[2] It is believed that this equation is ascribed to the Swiss mathematician Daniel Bernoulli only by inference, to give recognition to his pioneer achievement in hydrodynamics, in particular the introduction of the concept of "head." Actually, this equation was first formulated by Leonhard Euler and later popularized by Julius Weisbach [1].

3-2. Specific Energy. *Specific energy*[1] in a channel section is defined as the energy per pound of water at any section of a channel measured with respect to the channel bottom. Thus, according to Eq. (3-2) with $z = 0$, the specific energy becomes

$$E = d \cos \theta + \alpha \frac{V^2}{2g} \tag{3-7}$$

or, for a channel of small slope and $\alpha = 1$,

$$E = y + \frac{V^2}{2g} \tag{3-8}$$

which indicates that the specific energy is equal to the sum of the depth of water and the velocity head. For simplicity, the following discussion will be based on Eq. (3-8) for a channel of small slope. Since $V = Q/A$, Eq. (3-8) may be written $E = y + Q^2/2gA^2$. It can be seen that, for a given channel section and discharge Q, the specific energy in a channel section is a function of the depth of flow only.

When the depth of flow is plotted against the specific energy for a given channel section and discharge, a *specific-energy curve* (Fig. 3-2) is obtained. This curve has two limbs AC and BC. The limb AC approaches the horizontal axis asymptotically toward the right. The limb BC approaches the line OD as it extends upward and to the right. Line OD is a line that passes through the origin and has an angle of inclination equal to 45°. For a channel of large slope, the angle of inclination of the line OD will be different from 45°. (Why?) At any point P on this curve, the ordinate represents the depth, and the abscissa represents the specific energy, which is equal to the sum of the pressure head y and the velocity head $V^2/2g$.

The curve shows that, for a given specific energy, there are two possible depths, for instance, the *low stage* y_1 and the *high stage* y_2. The low stage is called the *alternate depth* of the high stage, and vice versa. At point C, the specific energy is a minimum. It will be proved later that this condition of minimum specific energy corresponds to the critical state of flow. Thus, at the critical state the two alternate depths apparently become one, which is known as the *critical depth* y_c. When the depth of flow is greater than the critical depth, the velocity of flow is less than the critical velocity for the given discharge, and, hence, the flow is subcritical. When the depth of flow is less than the critical depth, the flow is supercritical. Hence, y_1 is the depth of a supercritical flow, and y_2 is the depth of a subcritical flow.

If the discharge changes, the specific energy will be changed accordingly. The two curves $A'B'$ and $A''B''$ (Fig. 3-2) represent positions of the specific-energy curve when the discharge is less and greater, respectively, than the discharge used for the construction of the curve AB.

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[1] The concept of specific energy was first introduced by Bakhmeteff [2] in 1912.

3-3. Criterion for a Critical State of Flow. The critical state of flow has been defined (Art. 1-3) as the condition for which the Froude number is equal to unity. A more common definition is that it is the state of flow at which the specific energy is a minimum for a given discharge.[1] A theoretical criterion for critical flow may be developed from this definition as follows:

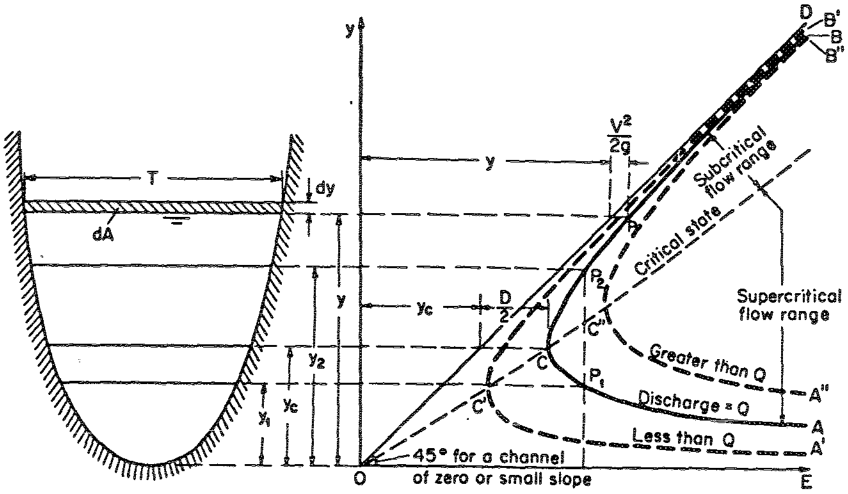


FIG. 3-2. Specific-energy curve.

Since $V = Q/A$, Eq. (3-8), the equation for specific energy in a channel of small slope with $\alpha = 1$, may be written

$$E = y + \frac{Q^2}{2gA^2} \tag{3-9}$$

Differentiating with respect to y and noting that Q is a constant,

$$\frac{dE}{dy} = 1 - \frac{Q^2}{gA^3}\frac{dA}{dy} = 1 - \frac{V^2}{gA}\frac{dA}{dy}$$

The differential water area dA near the free surface (Fig. 3-2) is equal to $T\,dy$. Now $dA/dy = T$, and the hydraulic depth $D = A/T$; so the above equation becomes

$$\frac{dE}{dy} = 1 - \frac{V^2T}{gA} = 1 - \frac{V^2}{gD}$$

---

[1] The concept of critical depth based on the theorem of minimum energy was first introduced by Böss [3].

At the critical state of flow the specific energy is a minimum, or $dE/dy = 0$. The above equation, therefore, gives

$$\frac{V^2}{2g} = \frac{D}{2} \tag{3-10}$$

This is the criterion for critical flow, which states that *at the critical state of flow, the velocity head is equal to half the hydraulic depth.* The above equation may also be written $V/\sqrt{gD} = 1$, which means $\mathbf{F} = 1$; this is the definition of critical flow given previously (Art. 1-3).

If the above criterion is to be used in any problem, the following conditions must be satisfied: (1) flow parallel or gradually varied, (2) channel of small slope, and (3) energy coefficient assumed to be unity. If the energy coefficient is not assumed to be unity, the critical-flow criterion is

$$\alpha \frac{V^2}{2g} = \frac{D}{2} \tag{3-11}$$

For a channel of large slope angle θ and energy coefficient α, the criterion for critical flow can easily be proved to be

$$\alpha \frac{V^2}{2g} = \frac{D \cos \theta}{2} \tag{3-12}$$

where D is the hydraulic depth of the water area normal to the channel bottom. In this case, the Froude number may be defined as

$$\mathbf{F} = \frac{V}{\sqrt{gD \cos \theta / \alpha}} \tag{3-13}$$

It should be noted that the coefficient α of a channel section actually varies with depth. In the above derivation, however, the coefficient is assumed to be constant; therefore, the resulting equation is not absolutely exact.

3-4. Interpretation of Local Phenomena. Change of the state of flow from subcritical to supercritical or vice versa occurs frequently in open channels. Such change is manifested in a corresponding change in the depth of flow from a high stage to a low stage or vice versa. If the change takes place rapidly over a relatively short distance, the flow is rapidly varied, and is known as a *local phenomenon.* The hydraulic drop and hydraulic jump are the two types of local phenomena, and may be described as follows:

Hydraulic Drop. A rapid change in the depth of flow from a high stage to a low stage will result in a steep depression in the water surface. Such a phenomenon is generally caused by an abrupt change in the channel slope or cross section and is known as a *hydraulic drop* (Fig. 1-2). At the transitory region of the hydraulic drop a reverse curve usually appears, connecting the water surfaces before and after the drop. The point of inflection on the reverse curve marks the approximate position of the critical depth at which the specific energy is a minimum and the flow passes from a subcritical state to a supercritical state.

The *free overfall* (Fig. 3-3) is a special case of the hydraulic drop. It occurs where the bottom of a flat channel is discontinued. As the free overfall enters the air in the form of a nappe, there will be no reverse curve in the water surface until it strikes some object at a lower elevation. It is the law of nature that, if no energy were added from the outside, the water surface would seek its lowest possible position corresponding to the least possible content of energy dissipation. If the specific energy at an upstream section is E, as shown on the specific-energy curve, it will continue to be dissipated on the way downstream and will finally reach a minimum energy content E_{min}. The specific-energy curve shows that the section of minimum energy or the critical section should occur at the brink. The brink depth cannot be less than the critical depth because further decrease in depth would require an increase in specific energy, which is impossible unless compensating external energy is supplied. The theoretical water-surface curve of an overfall is shown with a dashed line in Fig. 3-3.

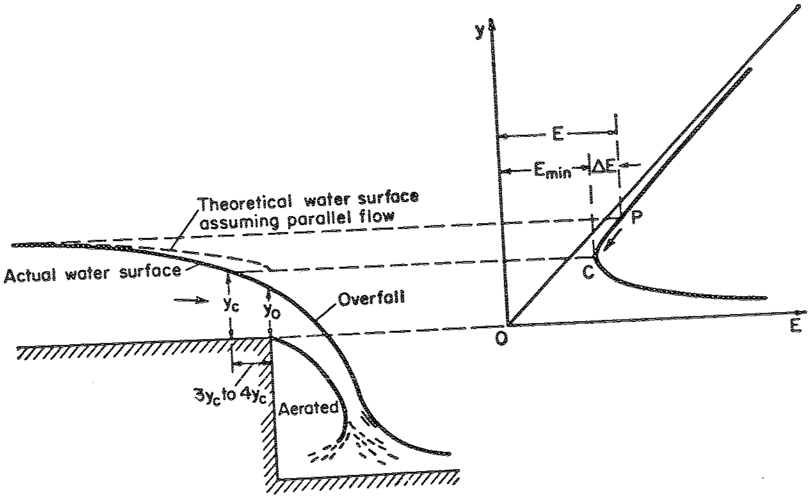
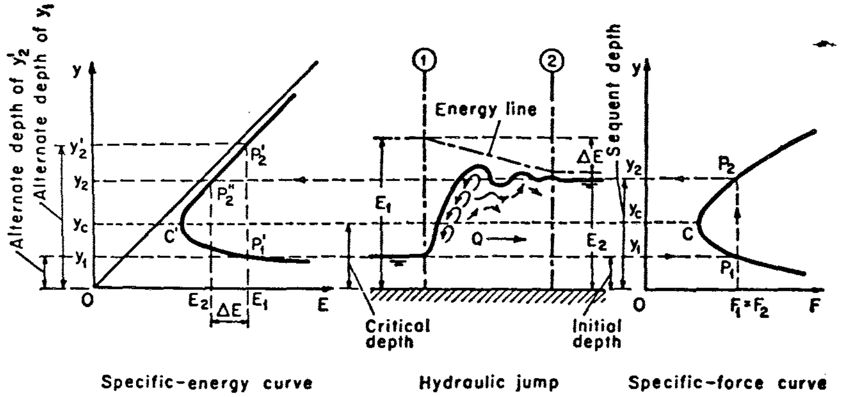


FIG. 3-3. Free overfall interpreted by specific-energy curve.

It should be remembered that the determination of critical depth by Eq. (3-10) or (3-11) is based on the assumption of parallel flow and is applicable only approximately to gradually varied flow. The flow at the brink is actually curvilinear, for the curvature of flow is pronounced; hence, the method is invalid for determining the critical depth as the depth at the brink. The actual situation is that the brink section is the *true* section of minimum energy, but it is not the critical section as computed by the principle based on the parallel-flow assumption. Rouse [4] found that for small slopes the computed critical depth is about 1.4 times the brink depth, or $y_c = 1.4y_0$, and that it is located about $3y_c$ to $4y_c$ behind the brink in the channel. The actual water surface of the overfall is shown by the full line (Fig. 3-3).

It should be noted that, if the change in the depth of flow from a high stage to a low stage is gradual, the flow becomes a gradually varied flow having a prolonged reversed curve of water surface; this phenomenon may be called a *gradual hydraulic drop* and is no longer a local phenomenon.



Specific-energy curve Hydraulic jump Specific-force curve

FIG. 3-4. Hydraulic jump interpreted by specific-energy and specific-force curves.

Hydraulic Jump. When the rapid change in the depth of flow is from a low stage to a high stage, the result is usually an abrupt rise of water surface (Fig. 3-4, in which the vertical scale is exaggerated). This local phenomenon is known as the *hydraulic jump.* It occurs frequently in a canal below a regulating sluice, at the foot of a spillway, or at the place where a steep channel slope suddenly turns flat.

If the jump is low, that is, if the change in depth is small, the water will not rise obviously and abruptly but will pass from the low to the high stage through a series of undulations gradually diminishing in size. Such a low jump is called an *undular jump.*

When the jump is high, that is, when the change in depth is great, the jump is called a *direct jump.* The direct jump involves a relatively large amount of energy loss through dissipation in the turbulent body of water in the jump. Consequently, the energy content in the flow after the jump is appreciably less than that before the jump.

It may be noted that the depth before the jump is always less than the depth after the jump. The depth before the jump is called the *initial depth* y_1 and that after the jump is called the *sequent depth* y_2. The initial and sequent depths y_1 and y_2 are shown on the specific-energy curve (Fig. 3-4). They should be distinguished from the alternate depths y_1 and y_2', which are the two possible depths for the same specific energy. The initial and sequent depths are actual depths before and after a jump in which an energy loss ΔE is involved. In other words, the specific energy E_1 at the initial depth y_1 is greater than the specific energy E_2 at the sequent depth y_2 by an amount equal to the energy loss ΔE. If there were no energy losses, the initial and sequent depths would become identical with the alternate depths in a prismatic channel.

3-5. Energy in Nonprismatic Channels. In preceding discussions the channel has been assumed prismatic so that one specific-energy curve could be applied to all sections of the channel. For nonprismatic channels, however, the channel section varies along the length of the channel and, hence, the specific-energy curve differs from section to section. This complication can be seen in a three-dimensional plot of the energy curves along the given reach of a nonprismatic channel.

For demonstrative purposes, a nonprismatic channel with variable slope is taken as an example, in which a gradually varied flow is carried from a subcritical state to a supercritical state (Fig. 3-5). The vertical profile of the channel along its center line is plotted on the Hx plane with the x axis chosen as the datum. For a variable-slope channel, it is more convenient to plot the total energy head $H = z + y + V^2/2g$, instead of the specific energy, against the depth of flow on the Hy plane. For simplicity, the pressure correction due to the slope angle and curvature of flow is ignored in this discussion. An energy line is then plotted on the Hx plane below a line parallel to the x axis and passing through the initial total head at the H axis. The exact position of the energy line depends on the energy losses along the channel. Four channel sections are then selected, and four energy curves for these sections are plotted in the Hy planes, as shown. The initial section 0 is an upstream section in the subcritical-flow region. The two depths corresponding to a given total energy H_0 can be obtained from the energy curve. Since this section is in the subcritical-flow region, the high stage y_0 should be the actual depth of flow, whereas the low stage is the alternate depth. Similarly, the alternate depths in other sections can be obtained. In the downstream sections 1 and 2, the low stages y_1 and y_2 are the actual depths of flow since they are in the supercritical-flow region. The critical depth at each section can also be obtained from the energy curve at the point of minimum energy. At section C the critical flow occurs, and the depth y_c is the critical depth. On the Hx plane, various lines can finally be plotted, showing the channel bottom, water surface, critical-depth line, and critical depth at this section is equal to the total head divided by 1.5 (Prob. 3-3), or 5.062/1.5 = 3.375 ft. By Eq. (3-10), the critical velocity is equal to $V_c = \sqrt{3.375g} = 10.45$ fps. Hence, the width of this critical section should be $100/(10.45 \times 3.38) = 2.83$ ft.

With the size of the mid-section determined, the side walls of the contraction can be drawn in with straight lines. The low and high stages at each section are then computed by the equation previously given. As the flow upstream from the critical section is subcritical, its water surface should follow the high stage. Downstream from the critical section, the flow is supercritical and its surface profile follows the low-stage line.

The critical-depth line is shown to separate the high from the low stage or the subcritical from the supercritical region of flow. On the basis of Eq. (3-10), the critical depth can be computed from the equation

$$\frac{(100/by_c)^2}{2g} = \frac{y_c}{2}$$

or

$$y_c = \sqrt[3]{\frac{10{,}000}{gb^2}}$$

where b is the width of the channel, which can be measured from the plan.

It should be noted that the vertical scale of the channel profile is greatly exaggerated. Furthermore, the outline of the gradual hydraulic drop is only theoretical, based on the theory of parallel flow. In reality, the flow near the drop is more or less curvilinear, and the actual profile would deviate from the theoretical one.

This example also serves to demonstrate a method of designing a channel transition (Arts. 11-5 to 11-7). The designer may fit any type of contraction walls he desires to suit a given flow profile, or vice versa.

3-6. Momentum in Open-channel Flow. As stated earlier (Art. 2-7), the momentum of the flow passing a channel section per unit time is expressed by $\beta wQV/g$, where β is the momentum coefficient, w is the unit weight of water in lb/ft³, Q is the discharge in cfs, and V is the mean velocity in fps.

According to Newton's second law of motion, the change of momentum per unit of time in the body of water in a flowing channel is equal to the resultant of all the external forces that are acting on the body. Applying this principle to a channel of large slope (Fig. 3-7), the following expression for the momentum change per unit time in the body of water enclosed between sections 1 and 2 may be written:

$$\frac{Qw}{g}(\beta_2 V_2 - \beta_1 V_1) = P_1 - P_2 + W \sin\theta - F_f \qquad (3\text{-}14)$$

where Q, w, and V are as previously defined, with subscripts referring to sections 1 and 2; P_1 and P_2 are the resultants of pressures acting on the two sections; W is the weight of water enclosed between the sections; and F_f is the total external force of friction and resistance acting along the surface of contact between the water and the channel. The above equation is known as the *momentum equation.*[1]

---

[1] The application of the momentum principle was first suggested by Bélanger [5].

For a parallel or gradually varied flow, the values of P_1 and P_2 in the momentum equation may be computed by assuming a hydrostatic distribution of pressure. For a curvilinear or rapidly varied flow, however, the pressure distribution is no longer hydrostatic; hence the values of P_1 and P_2 cannot be so computed but must be corrected for the curvature effect of the streamlines of the flow. For simplicity, P_1 and P_2 may be replaced, respectively, by $\beta_1'P_1$ and $\beta_2'P_2$, where β_1' and β_2' are the correction coefficients at the two sections. The coefficients are referred to as *pressure-distribution coefficients*. Since P_1 and P_2 are forces, the coefficients may be specifically called *force coefficients*. It can be shown that the force coefficient is expressed by

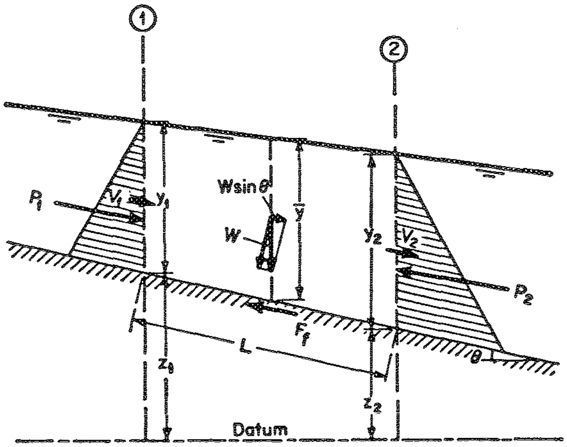


FIG. 3-7. Application of the momentum principle.

$$\beta' = \frac{1}{A\bar{z}} \int_0^A h\,dA = 1 + \frac{1}{A\bar{z}} \int_0^A c\,dA \tag{3-15}$$

where $\bar{z}$ is the depth of the centroid of the water area A below the free surface, h is the pressure head on the elementary area dA, and c is the pressure-head correction [Eq. (2-9)]. It can easily be seen that β' is greater than 1.0 for concave flow, less than 1.0 for convex flow, and equal to 1.0 for parallel flow.

It can be shown that the momentum equation is similar to the energy equation when applied to certain flow problems. In this case, a gradually varied flow is considered; accordingly, the pressure distribution in the sections may be assumed hydrostatic, and $\beta' = 1$. Also, the slope of the channel is assumed relatively small.[1] Thus, in the short reach of a rectangular channel of small slope and width b (Fig. 3-7),

---

[1] If the slope angle θ is large, then $P_1 = \frac{1}{2}wd_1^2 \cos\theta$ and $P_2 = \frac{1}{2}wd_2^2 \cos\theta$, where d_1 and d_2 are the depths of flow section and $\cos\theta$ is a correction factor (Art. 2-10).

$$P_1 = \tfrac{1}{2}wby_1^2$$

and

$$P_2 = \tfrac{1}{2}wby_2^2$$

Assume

$$F_f = wh_f'b\bar{y}$$

where h_f' is the friction head and $\bar{y}$ is the average depth, or $(y_1 + y_2)/2$. The discharge through the reach may be taken as the product of the average velocity and the average area, or

$$Q = \tfrac{1}{2}(V_1 + V_2)b\bar{y}$$

Also, it is evident (Fig. 3-7) that the weight of the body of water is

$$W = wb\bar{y}L$$

and

$$\sin\theta = \frac{z_1 - z_2}{L}$$

Substituting all the above expressions for the corresponding items in Eq. (3-14) and simplifying,

$$z_1 + y_1 + \beta_1\frac{V_1^2}{2g} = z_2 + y_2 + \beta_2\frac{V_2^2}{2g} + h_f' \qquad (3\text{-}16)$$

This equation appears to be practically the same as the energy equation (3-5).

Theoretically speaking, however, the two equations not only use different velocity-distribution coefficients, although these are nearly equal, but also involve different meanings of the frictional losses. In the energy equation, the item h_f measures the *internal* energy dissipated in the whole mass of the water in the reach, whereas the item h_f' in the momentum equation measures the losses due to *external* forces exerted on the water by the walls of the channel. Ignoring the small difference between the coefficients α and β, it seems that, in gradually varied flow, the internal-energy losses are practically identical with the losses due to external forces. In uniform flow, the rate with which surface forces are doing work is equal to the rate of energy dissipation. In that case, therefore, a distinction between h_f and h_f' does not exist except in definition.

The similarity between the applications of the energy and momentum principles may be confusing. A clear understanding of the basic differences in their constitution is important, despite the fact that in many instances the two principles will produce practically identical results. The inherent distinction between the two principles lies in the fact that energy is a scalar quantity whereas momentum is a vector quantity; also, the energy equation contains a term for internal losses, whereas the momentum equation contains a term for external resistance.

Generally speaking, the energy principle offers a simpler and clearer explanation than does the momentum principle. But the momentum principle has certain advantages in application to problems involving high internal-energy changes, such as the problem of the hydraulic jump. If the energy equation is applied to such problems, the unknown internal-energy loss represented by h_f is indeterminate, and the omission of this term would result in considerable errors. If instead the momentum equation is applied to these problems, since it deals only with external forces, the effects of the internal forces will be entirely out of consideration and need not be evaluated. The term for frictional losses due to external forces, on the other hand, is unimportant in such problems and can safely be omitted, because the phenomenon takes place in a short reach of the channel and the effect due to external forces is negligible compared with the internal losses. Further discussions on the solution of the hydraulic-jump problem by both principles will be given later (Example 3-3).

An example showing the application of the momentum principle to the problem of a broad-crested weir is given below.

Example 3-2. Derive the discharge per unit width of a broad-crested weir across a rectangular channel.

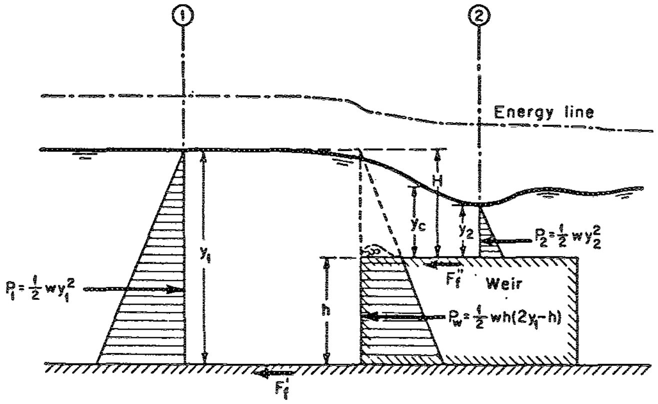

①
②
Energy line
H
y_c
y_2
$P_2 = \frac{1}{2} w y_2^2$
F_f''
Weir
y_1
$P_1 = \frac{1}{2} w y_1^2$
h
$P_w = \frac{1}{2} wh(2y_1 - h)$
F_f'

FIG. 3-8. Momentum principle applied to flow over a broad-crested weir.

Solution. The assumptions to be made in this solution (Fig. 3-8) are (1) the frictional forces F_f' and F_f'' are negligible; (2) the depth y_2 is the minimum depth on the weir; (3) at the channel sections under consideration there is parallel flow; and (4) the water pressure P_w on the weir surface is equal to the total hydrostatic pressure measured below the upstream water surface, or

$$P_w = \tfrac{1}{2}wh[y_1 + (y_1 - h)] = \tfrac{1}{2}wh(2y_1 - h)$$

The accuracy of the last assumption has been checked experimentally [6]. If the momentum equation (3-14) is applied to the body of water between the upstream approach section 1 and the downstream section 2 at the minimum depth on the top of the weir, the following equation may be written:

$$\frac{qw}{g}\left(\frac{q}{y_2} - \frac{q}{y_1}\right) = \tfrac{1}{2}wy_1^2 - \tfrac{1}{2}wy_2^2 - \tfrac{1}{2}wh(2y_1 - h)$$

where q is the discharge per unit width of the weir.

Experiments by Doeringsfeld and Barker [6] have shown that, on the average, $y_1 - h = 2y_2$. In that case the above equation can be simplified and solved for q,

$$q = 0.433\sqrt{2g}\left(\frac{y_1}{y_1 + h}\right)^{1/2} H^{3/2} \tag{3-17}$$

Considering the limit of h from zero to infinity, this equation varies from $q = 3.47H^{3/2}$ to $q = 2.46H^{3/2}$. It is interesting to note that the practical range of the coefficient to $H^{3/2}$ obtained by actual observations[1] is from 3.05 to 2.67. In applying the momentum principle to this problem, it can be seen that knowledge of the internal-energy losses due to separation of flow at the entrance and to other causes is not needed in the analysis.

3-7. Specific Force. In applying the momentum principle to a short horizontal reach of a prismatic channel, the external force of friction and the weight effect of water can be ignored. Thus, with $\theta = 0$ and $F_f = 0$ and assuming also $\beta_1 = \beta_2 = 1$, Eq. (3-14) becomes

$$\frac{Qw}{g}(V_2 - V_1) = P_1 - P_2$$

The hydrostatic forces P_1 and P_2 may be expressed as

$$P_1 = w\bar{z}_1A_1 \qquad \text{and} \qquad P_2 = w\bar{z}_2A_2$$

where $\bar{z}_1$ and $\bar{z}_2$ are the distances of the centroids of the respective water areas A_1 and A_2 below the surface of flow. Also, $V_1 = Q/A_1$ and $V_2 = Q/A_2$. Then, the above momentum equation may be written

$$\frac{Q^2}{gA_1} + \bar{z}_1A_1 = \frac{Q^2}{gA_2} + \bar{z}_2A_2 \tag{3-18}$$

---

[1] The value of the coefficient actually depends on many factors: mainly, the rounding of the upstream corner, the length and slope of the weir crest, and the height of the weir. Many experiments on *broad-crested weirs* have been performed. From several of the well-known experiments King [7] has interpolated the data and prepared tables for the coefficient under various conditions. A comprehensive analysis including more recent data and a presentation of the results for practical applications were made by Tracy [8]. The well-known experiments on broad-crested weirs are (1) *Bazin tests* performed in Dijon, France, in 1886 [9]; (2) *U.S.D.W.B. Cornell tests* performed at Cornell University in 1899 by the U.S. Deep Waterways Board under the direction of G. W. Rafter, and *U.S.G.S. Cornell tests* performed by the U.S. Geological Survey under the direction of Robert E. Horton in 1903 [10]; (3) *Michigan tests* performed at the University of Michigan during 1928–1929 [11]; and (4) *Minnesota* and *Washington tests* performed, respectively, at the university of Minnesota and Washington State University [6]. For some formulas and coefficients of discharge developed in the U.S.S.R., see [12]. For an analytical treatment of the problem, see [13].

The two sides of Eq. (3-18) are analogous and, hence, may be expressed for any channel section by a general function

$$F = \frac{Q^2}{gA} + \bar{z}A \tag{3-19}$$

This function consists of two terms. The first term is the momentum of the flow passing through the channel section per unit time per unit weight of water, and the second is the force per unit weight of water. Since both terms are essentially force per unit weight of water, their sum may be called the *specific force*.[1] Accordingly, Eq. (3-18) may be expressed as $F_1 = F_2$. This means that the specific forces of sections 1 and 2 are equal, provided that the external forces and the weight effect of water in the reach between the two sections can be ignored.

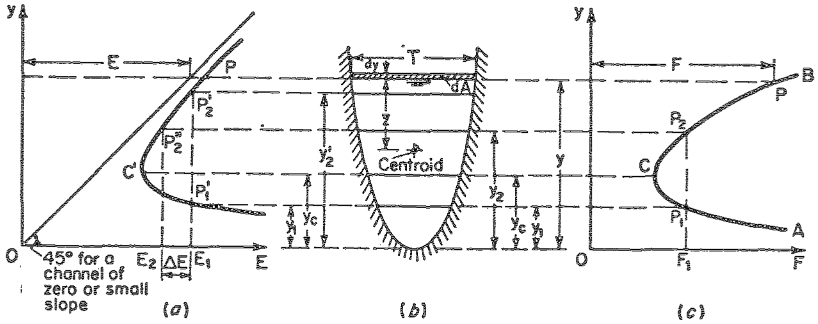


FIG. 3-9. Specific-force curve supplemented with specific-energy curve. (*a*) Specific-energy curve; (*b*) channel section; (*c*) specific-force curve.

By plotting the depth against the specific force for a given channel section and discharge, a *specific-force curve* is obtained (Fig. 3-9). This curve has two limbs AC and BC. The limb AC approaches the horizontal axis asymptotically toward the right. The limb BC rises upward and extends indefinitely to the right. For a given value of the specific force, the curve has two possible depths y_1 and y_2. As will be shown later, the two depths constitute the initial and sequent depths of a hydraulic jump. At point C on the curve the two depths become one, and the specific force is a minimum. The following argument shows that the depth at the minimum value of the specific force is equal to the critical depth.[2]

---

[1] This has been variously called the "force plus momentum," the "momentum flux," the "total force," or, briefly, the "force" of a stream (see pp. 81 and 82 of [14]). The function represented by Eq. (3-19) was formulated by Bresse [15] for the study of the hydraulic jump to be described in Example 3-3.

[2] The concept of critical depth based on the theorem of momentum is believed to have been developed by Boussinesq [16].

For a minimum value of the specific force, the first derivative of F with respect to y should be zero, or, from Eq. (3-19),

$$\frac{dF}{dy} = -\frac{Q^2}{gA^2}\frac{dA}{dy} + \frac{d(\bar{z}A)}{dy} = 0$$

For a change dy in the depth, the corresponding change $d(\bar{z}A)$ in the static moment of the water area about the free surface is equal to $[A(\bar{z} + dy) + T(dy)^2/2] - \bar{z}A$ (Fig. 3-9). Ignoring the differential of higher degree, that is, assuming $(dy)^2 = 0$, the change in static moment becomes $d(\bar{z}A) = A\,dy$. Then the preceding equation may be written

$$\frac{dF}{dy} = -\frac{Q^2\,dA}{gA^2\,dy} + A = 0$$

Since $dA/dy = T$, $Q/A = V$, and $A/T = D$, the above equation may be reduced to

$$\frac{V^2}{2g} = \frac{D}{2} \tag{3-10}$$

This is the criterion for the critical state of flow, derived earlier (Art. 3-3). Therefore, it is proved that the depth at the minimum value of the specific force is the critical depth.[1] It may also be stated that *at the critical state of flow the specific force is a minimum for the given discharge.*

Now, compare the specific-force curve with the specific-energy curve (Fig. 3-9). For a given specific energy E_1, the specific-energy curve indicates two possible depths, namely, a low stage y_1 in the supercritical flow region and a high stage y_2' in the subcritical flow region.[2] For a given value of F_1, the specific-force curve also indicates two possible depths, namely, an initial depth y_1 in the supercritical region and a sequent depth y_2 in the subcritical flow region. It is assumed that the low stage and the initial depth are both equal to y_1. Thus, the two curves indicate jointly that the sequent depth y_2 is always less than the high stage y_2'. Furthermore, the specific-energy curve shows that the energy content E_2 for the depth y_2 is less than the energy content E_1 for the depth y_2'. Therefore, in order to maintain a constant value of F_1, the depth of flow may be changed from y_1 to y_2 at the price of losing a certain amount of energy, which is equal to $E_1 - E_2 = \Delta E$. One example of this is the hydraulic jump on a horizontal floor, in which the specific forces before and after the jump are equal and the loss of energy is a consequence of the phenomenon. This will be explained further in the following example. It may be noted at this point, however, that the depths y_1 and y_2' shown by the specific-energy curve are the alternate depths; whereas the depths y_1 and y_2 shown by the specific-force curve are, respectively, the initial depth and the sequent depth of a hydraulic jump.

---

[1] It should be noted that the above proof is based on the assumptions of parallel flow and uniform velocity distribution. However, the concept of critical depth is a general concept that is valid for all flows, whether derived from energy or from momentum considerations. This validity has been proved by Jaeger [14,17,18], and the proof is known as the *Jaeger theorem* [19].

[2] In order to make a clear distinction between the sequent depth and the high stage of the alternate depths, the sequent depth is designated by y_2 and the high stage by y_2'. In some other places in this book, however, both are designated by y_2.

Example 3-3. Derive a relationship between the initial depth and the sequent depth of a hydraulic jump on a horizontal floor in a rectangular channel.

Solution. The external forces of friction and the weight effect of water in the hydraulic jump on a horizontal floor are negligible, because the jump takes place in a relatively short distance and the slope angle of the horizontal floor is zero. The specific forces of sections 1 and 2 (Fig. 3-4), respectively, before and after the jump, can therefore be considered equal; that is,

$$\frac{Q^2}{gA_1} + \bar{z}_1 A_1 = \frac{Q^2}{gA_2} + \bar{z}_2 A_2 \qquad (3\text{-}18)$$

For a rectangular channel of width b, $Q = V_1 A_1 = V_2 A_2$, $A_1 = by_1$, $A_2 = by_2$, $\bar{z}_1 = y_1/2$, and $\bar{z}_2 = y_2/2$. Substituting these relations and $\mathbf{F}_1 = V_1/\sqrt{gy_1}$ in the above equation and simplifying,

$$\left(\frac{y_2}{y_1}\right)^3 - (2\mathbf{F}_1^2 + 1)\left(\frac{y_2}{y_1}\right) + 2\mathbf{F}_1^2 = 0 \qquad (3\text{-}20)$$

Factoring,

$$\left[\left(\frac{y_2}{y_1}\right)^2 + \frac{y_2}{y_1} - 2\mathbf{F}_1^2\right]\left(\frac{y_2}{y_1} - 1\right) = 0$$

Then, let

$$\left(\frac{y_2}{y_1}\right)^2 + \frac{y_2}{y_1} - 2\mathbf{F}_1^2 = 0$$

The solution of this quadratic equation is

$$\frac{y_2}{y_1} = \tfrac{1}{2}\left(\sqrt{1 + 8\mathbf{F}_1^2} - 1\right) \qquad (3\text{-}21)$$

For a given Froude number $\mathbf{F}_1$ of the approaching flow, the ratio of the sequent depth to the initial depth is given by the above equation.

It should be understood that the momentum principle is used in this solution because the hydraulic jump involves a high amount of internal-energy losses which cannot be evaluated in the energy equation.

The joint use of the specific-energy curve and the specific-force curve helps to determine graphically the energy loss involved in the hydraulic jump for a given approaching flow. For the given approaching depth y_1, points P_1 and P_1' are located on the specific-force curve and the specific-energy curve, respectively (Fig. 3-4). The point P_1' gives the initial energy content E_1. Draw a vertical line, passing through the point P_1 and intercepting the upper limb of the specific-force curve at point P_2, which gives the sequent depth y_2. Then, draw a horizontal line passing through the point P_2 and intercepting the specific-energy curve at point P_2'', which gives the energy content E_2 after the jump. The energy loss in the jump is then equal to $E_1 - E_2$, represented by ΔE.

3-8. Momentum Principle Applied to Nonprismatic Channels. The specific force, like the specific energy, varies with the shape of the channel

CHAPTER 4

CRITICAL FLOW: ITS COMPUTATION AND APPLICATIONS

4-1. Critical Flow. As described in the previous chapter, the critical state of flow through a channel section is characterized by several important conditions.[1] Recapitulating, they are (1) the specific energy is a minimum for a given discharge; (2) the discharge is a maximum for a given specific energy (Prob. 3-7); (3) the specific force is a minimum for a given discharge; (4) the velocity head is equal to half the hydraulic depth in a channel of small slope; (5) the Froude number is equal to unity; and (6) the velocity of flow in a channel of small slope with uniform velocity distribution is equal to the celerity of small gravity waves in shallow water caused by local disturbances.

Discussions on critical state of flow have referred mainly to a particular section of a channel, known as the *critical section*. If the critical state of flow exists throughout the entire length of the channel or over a reach of the channel, the flow in the channel is a *critical flow*. Since, as indicated by the critical-flow criterion Eq. (3-10), the depth of critical flow depends on the geometric elements A and D of the channel section when the discharge is constant, the critical depth in a prismatic channel of uniform slope will be the same in all sections, and critical flow in a prismatic channel should, therefore, be uniform flow. At this condition, the slope of the channel that sustains a given discharge at a uniform and critical depth is called the *critical slope* S_c. A slope of the channel less than the critical slope will cause a slower flow of subcritical state for the given discharge, as will be shown later, and, hence, is called a *mild* or *subcritical slope*. A slope greater than the critical slope will result in a faster flow of supercritical state, and is called a *steep* or *supercritical slope*.

A flow at or near the critical state is unstable. This is because a minor change in specific energy at or close to critical state will cause a major change in depth. This fact can also be recognized in the specific-energy curve (Fig. 3-2). As the curve is almost vertical near the critical depth, a slight change in energy would change the depth to a much smaller or much greater alternate depth corresponding to the specific energy after the change. It can be observed also that, when the flow is near the critical state, the water surface appears unstable and wavy. Such phenomena are generally caused by the minor changes in energy due to variations in channel roughness, cross section, slope, or deposits of sediment or debris. In the design of a channel, if the depth is found at or near the critical depth for a great length of the channel, the shape or slope of the channel should be altered, if practicable, in order to secure greater stability.

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[1] For a historical account of the theory of critical flow, see [1].

The criterion for a critical state of flow (Art. 3-3) is the basis for the computation of critical flow, which will be explained in subsequent articles. Two major applications of critical-flow theory are flow control and flow measurement, which will also be discussed in this chapter.

4-2. The Section Factor for Critical-flow Computation. Substituting $V = Q/A$ in Eq. (3-10) and simplifying,

$$Z = \frac{Q}{\sqrt{g}} \tag{4-1}$$

When the energy coefficient is not assumed to be unity,

$$Z = \frac{Q}{\sqrt{g/\alpha}} \tag{4-2}$$

In the above equations, $Z = A\sqrt{D}$, which is the *section factor for critical-flow computation* [Eq. (2-3)]. Equation (4-2) states that the section factor Z for a channel section at the critical state of flow is equal to the discharge divided by the square root of g/α. Since the section factor Z is generally a single-valued function of the depth, the equation indicates that there is *only one* possible critical depth for maintaining the given discharge in a channel and similarly that, when the depth is fixed, there can be *only one* discharge that maintains a critical flow and makes the depth critical in the given channel section.[1]

Equation (4-1) or (4-2) is a very useful tool for the computation and analysis of critical flow in an open channel. When the discharge is given, the equation gives the critical section factor Z_c and, hence, the critical depth y_c. On the other hand, when the depth and, hence, the section factor are given, the critical disharge can be computed by Eq. (4-1) in the following form:

$$Q = Z\sqrt{g} \tag{4-3}$$

or by Eq. (4-2) in the following form:

$$Q = Z\sqrt{\frac{g}{\alpha}} \tag{4-4}$$

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[1] If Z is not a single-valued function of the depth, it is possible to have more than one critical depth.

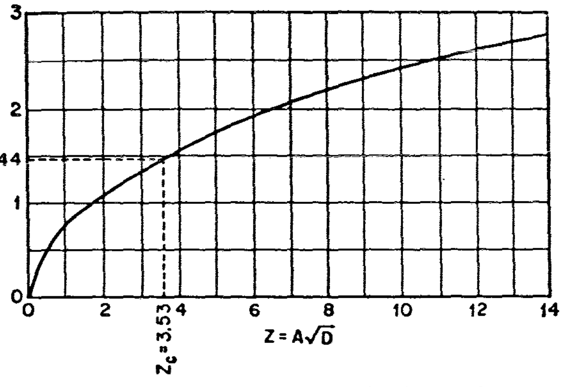

FIG. 4-4. Curve of y versus Z for a circular section.

In Example 4-3, $Z/d_0^{2.5} = 0.226$. For this value the chart gives $y/d = 0.48$ or $y_c = 1.44$ ft.

4-5. Control of Flow. The control of flow in an open channel is defined loosely in many ways. As used here the term means the establishment of a definitive flow condition in the channel or, more specifically, a definitive relationship between the stage and the discharge of the flow. When the control of flow is achieved at a certain section of the channel, this section is a *control section.* It will be shown later that the control section controls the flow in such a way that it restricts the transmission of the effect of changes in flow condition either in an upstream direction or in a downstream direction depending on the state of flow in the channel. Since the control section holds a definitive stage-discharge relationship, it is always a suitable site for a gaging station and for developing the *discharge rating curve,* a curve representing the depth-discharge relationship at the gaging station.

At the critical state of flow a definitive stage-discharge relationship can be established and represented by Eq. (4-1). This equation shows that the stage-discharge relationship is theoretically independent of the channel roughness and other uncontrolled circumstances. Therefore, a critical-flow section is a control section.

The location of the control section in a prismatic channel is generally governed by the state of flow, which in turn is determined by the slope of the channel. Take for an example a long straight prismatic channel in which a pool is created by a dam across the channel and the water flows over the dam through an overflow spillway (Fig. 4-5). Three flow conditions in the channel are shown, representing the subcritical, critical, and supercritical flows, respectively. The slopes of the channel in the three cases are, correspondingly, *mild* or subcritical, critical, and *steep* or supercritical.

If the channel has a critical slope (middle sketch in Fig. 4-5), then the flow is initially uniform and critical throughout the channel. In the presence of the dam, however, the flow through the pool will be subcritical and the pool surface will approach the horizontal. At the downstream end a so-called *drawdown curve* will be developed, extending upstream from a section near the spillway crest and becoming asymptotic to the pool level.

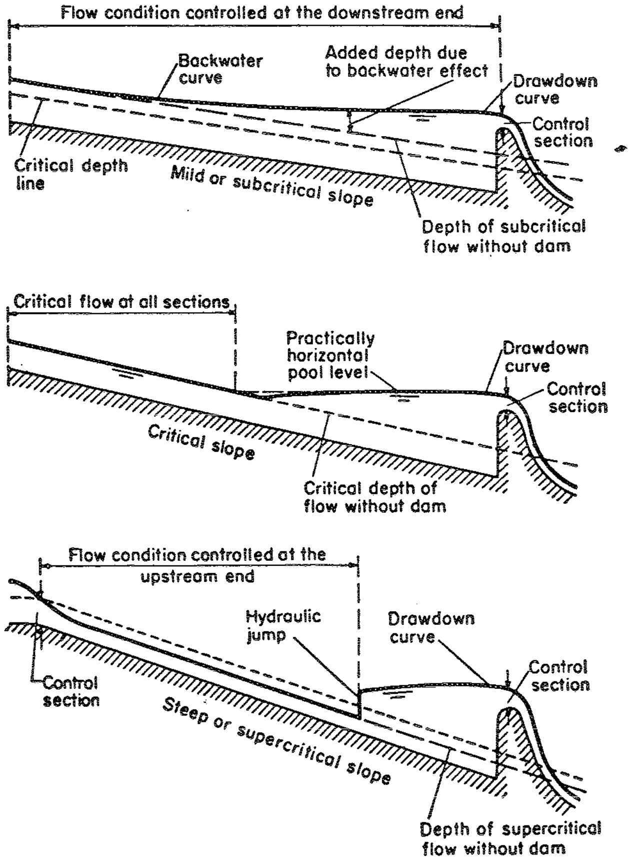
Flow condition controlled at the downstream end
Backwater curve
Added depth due to backwater effect
Drawdown curve
Control section
Critical depth line
Mild or subcritical slope
Depth of subcritical flow without dam
Critical flow at all sections
Practically horizontal pool level
Drawdown curve
Control section
Critical slope
Critical depth of flow without dam
Flow condition controlled at the upstream end
Hydraulic jump
Drawdown curve
Control section
Control section
Steep or supercritical slope
Depth of supercritical flow without dam

FIG. 4-5. Flow conditions in a long prismatic channel.

If the channel has a subcritical slope (top sketch in Fig. 4-5), the flow is initially subcritical. In the presence of the dam, the pool surface will be further raised for a long distance upstream from the pool in a so-called *backwater curve*. The additional depth of water is required to build up enough head to give the increased velocity necessary to pass water over the spillway. This effect of backing up the water behind the dam is known as the *backwater effect.* At the downstream end the backwater curve is connected with a smooth drawdown curve which leads the water over the spillway.

If the channel has a supercritical slope (bottom sketch in Fig. 4-5), the flow is initially supercritical. In the presence of the dam, the backwater effect originating from the pool will not extend far upstream. Instead, the flow in the upstream channel will continue in the downstream direction at a supercritical state until the flow-surface profile is actually below the pool level;[1] then it will rise abruptly to the pool elevation in a hydrau-

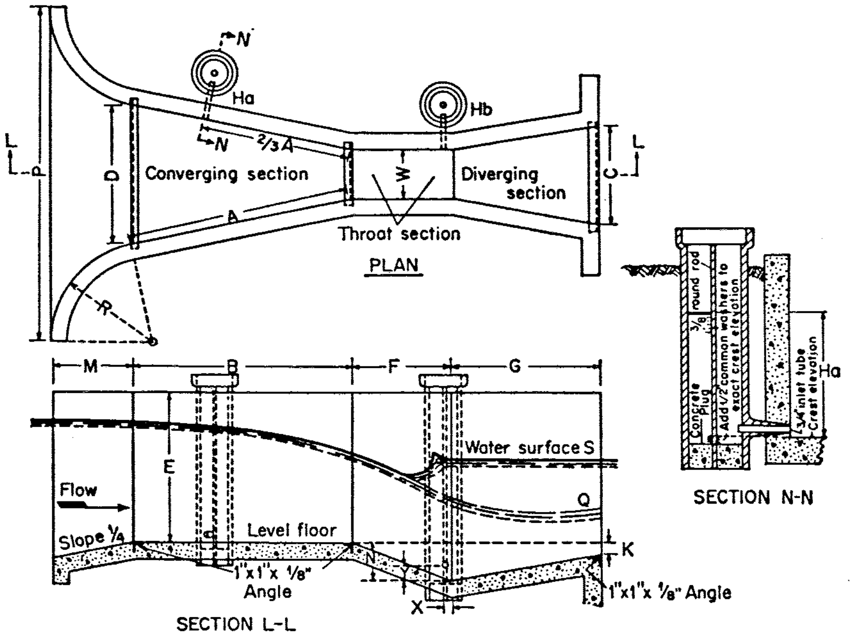

FIG. 4-6. Plan, elevation, and dimensions of the Parshall flume. (*U.S. Soil Conservation Service* [26].) Plan and elevation of a concrete Parshall measuring flume showing lettered dimensions as follows:

W = size of flume in in. or ft; A = length of side wall of converging section; $\frac{2}{3}A$ = distance back from end of crest to gage point; B = axial length of converging section; C = width of downstream end of flume; D = width of upstream end of flume; E = depth of flume; F = length of throat; G = length of diverging section; K = difference in elevation between lower end of flume and crest; M = length of approach floor; N = depth of depression in throat below crest; P = width between ends of curved wing walls; R = radius of curved wing wall; X = horizontal distance to H_b gage point from low point in throat; Y = vertical distance to H_b gage point from low point in throat. See the table on the next page for actual dimensions for various sizes of flume.

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[1] It should be noted that the pool level in this case is not horizontal but curved. The curved water surface has an S1 profile, which will be described later (Art. 9-4).

lic jump. The backwater effect will not extend upstream through the hydraulic jump. The flow upstream from the jump is governed entirely by the upstream conditions.

The above example explains the important fact that on subcritical slopes the effect of change in water-surface elevation downstream is transmitted upstream by a backwater curve, whereas on supercritical slopes the effect cannot be transmitted far upstream. The flow condition in a subcritical channel is affected by downstream conditions; but, in a supercritical channel, the flow condition is dependent entirely upon the condition upstream or at the place where water enters the channel. Accordingly, the control of flow is said to be at the downstream end for channels with subcritical slope and at the upstream end for channels with supercritical slope.

When the channel is on a subcritical slope a control section at the downstream end may be a critical section, such as that created on the top of an overflow spillway. On a supercritical slope, the control section at the upstream end may also be a critical section, as shown in the figure. A sluice gate or an orifice or other control structure may also be used to create a control section. It should be noted that whether the channel slope is critical, subcritical, or supercritical will depend not only on the measure of the actual slope but also on the discharge or the depth of flow.

4-6. Flow Measurement. It was mentioned in the preceding article that, at a critical control section, the relationship between the depth and the discharge is definitive, independent of the channel roughness and other uncontrollable circumstances. Such a definitive stage-discharge relationship offers a theoretical basis for the measurement of discharge in open channels.

Based on the principle of critical flow, various devices for flow measurement have been developed. In such devices the critical depth is usually created either by the construction of a low hump on the channel bottom, such as a weir, or by a contraction in the cross section, such as a *critical-flow flume*. The use of a weir is a simple method, but it causes relatively high head loss. If water contains suspended particles, some will be deposited in the upstream pool formed by the weir, resulting in a gradual change in the discharge coefficient. These difficulties, however, can be overcome at least partially by the use of a critical-flow flume.

The critical-flow flume, also known as the *Venturi flume*, has been designed in various forms.[1] It is usually operated with an unsubmerged

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[1] The critical-flow flumes mentioned in the text are those developed and studied in the United States. Outstanding designs of critical-flow flumes were also developed and tested by Jameson [4,5], Engel [6,7], and Linford [8] in England; by Crump [9] and Inglis [10] in India; by De Marchi [11,12], Contessini [11], Nebbia [13–15], and Citrini [16,17] in Italy; by Khafagi [18] in Switzerland; and by Balloffet [19] in Argentina.

CHAPTER 5

DEVELOPMENT OF UNIFORM FLOW AND ITS FORMULAS

5-1. Qualifications for Uniform Flow. The uniform flow to be considered has the following main features: (1) the depth, water area, velocity, and discharge at every section of the channel reach are constant; and (2) the energy line, water surface, and channel bottom are all parallel; that is, their slopes are all equal, or $S_f = S_w = S_0 = S$. For practical purposes, the requirement of constant velocity may be liberally interpreted as the requirement that the flow possess a constant mean velocity. Strictly speaking, however, this should mean that the flow possesses a constant velocity at every point on the channel section within the uniform-flow reach. In other words, the velocity distribution across the channel section is unaltered in the reach. Such a stable pattern of velocity distribution can be attained when the so-called "boundary layer" is fully developed (Art. 8-1).

Uniform flow is considered to be steady only, since unsteady uniform flow is practically nonexistent. In natural streams, even steady uniform flow is rare, for rivers and streams in natural states scarcely ever experience a strict uniform-flow condition. Despite this deviation from the truth, the uniform-flow condition is frequently assumed in the computation of flow in natural streams. The results obtained from this assumption are understood to be approximate and general, but they offer a relatively simple and satisfactory solution to many practical problems.

As turbulent uniform flow is most commonly encountered in engineering problems, it will be discussed extensively in the following chapters. Laminar uniform flow has limited engineering applications, and will be described only in Art. 6-10.

It should be noted that uniform flow cannot occur at very high velocities, usually described as *ultrarapid*. This is because, when uniform flow reaches a certain high velocity, it becomes very unstable. At higher velocities the flow will eventually entrain air and become unsteady. The criterion for instability of uniform flow will be discussed in Art. 8-8.

5-2. Establishment of Uniform Flow. When flow occurs in an open channel, resistance is encountered by the water as it flows downstream.

This resistance is generally counteracted by the components of gravity forces acting on the body of the water in the direction of motion (Fig. 5-2). A uniform flow will be developed if the resistance is balanced by the gravity forces. The magnitude of the resistance, when other physical factors of the channel are kept unchanged, depends on the velocity of flow.

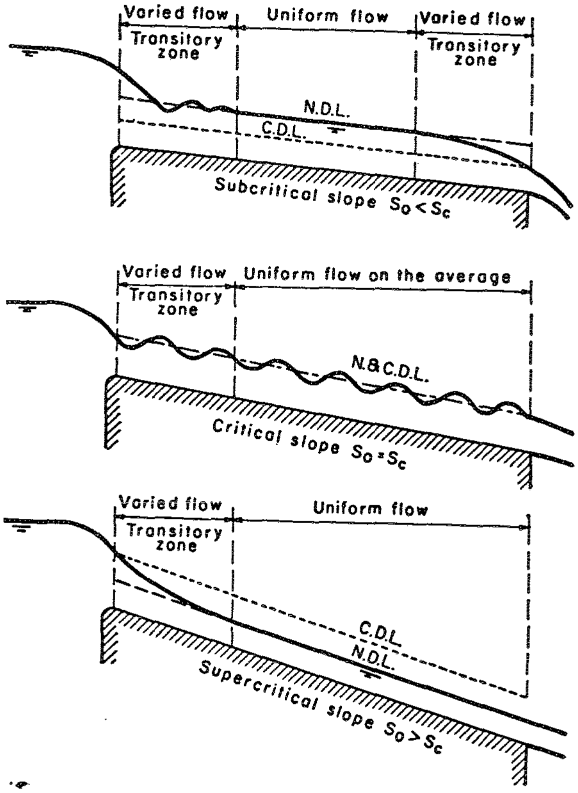

FIG. 5-1. Establishment of uniform flow in a long channel.

If the water enters the channel slowly, the velocity and hence the resistance are small, and the resistance is outbalanced by the gravity forces, resulting in an accelerating flow in the upstream reach. The velocity and the resistance will gradually increase until a balance between resistance and gravity forces is reached. At this moment and afterward the flow becomes uniform. The upstream reach that is required for the establishment of uniform flow is known as the *transitory zone*. In this zone the flow is accelerating and varied. If the channel is shorter than the transitory length required by the given conditions, uniform flow cannot be attained. Toward the downstream end of the channel the resistance may again be exceeded by gravity forces, and the flow may become varied again.

For purposes of explanation, a long channel is shown with three different slopes: subcritical, critical, and supercritical (Fig. 5-1). At the subcritical slope (top sketch in Fig. 5-1) the water surface in the transitory zone appears undulatory. The flow is uniform in the middle reach of the channel but varied at the two ends.[1] At the critical slope (middle sketch in Fig. 5-1) the water surface of the critical flow is unstable. Possible undulations may occur in the middle reach, but on the average the depth is constant and the flow may be considered uniform. At the supercritical slope (bottom sketch in Fig. 5-1) the transitory water surface passes from the subcritical stage to the supercritical stage through a gradual hydraulic drop. Beyond the transitory zone the flow is approaching uniformity. The depth of a uniform flow is called the *normal depth*. In all figures the long dashed line represents the normal-depth line, abbreviated as N.D.L., and the short dashed or dotted line represents the critical-depth line, or C.D.L.

The length of the transitory zone depends on the discharge and on the physical conditions of the channel, such as entrance condition, shape, slope, and roughness. From a hydrodynamic standpoint (see Art. 8-1), the length of the transitory zone should not be less than the length required for the full development of the boundary layer under the given conditions.

5-3. Expressing the Velocity of a Uniform Flow. For hydraulic computations the mean velocity of a turbulent uniform flow in open channels is usually expressed approximately by a so-called *uniform-flow formula*. Most practical uniform-flow formulas can be expressed in the following general form:

$$V = CR^xS^y \tag{5-1}$$

where V is the mean velocity in fps; R is the hydraulic radius in ft; S is the energy slope,[2] x and y are exponents; and C is a factor of flow resistance, varying with the mean velocity, hydraulic radius, channel roughness, viscosity, and many other factors.

For practical purposes, the flow in a natural channel may be assumed uniform under normal conditions, that is, if there are no flood flows or markedly varied flows caused by channel irregularities. In applying the uniform-flow formula to a natural stream, it is understood that th result is very approximate since the flow condition is subject to mor uncertain factors than would be involved in a regular artificial channe As pointed out by Schneckenberg [1], a good uniform-flow formula fo an alluvial channel with sediment transport and turbulent flow shoul take equal account of all the following variables:

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[1] Theoretically speaking, the varied depth at each end approaches the uniform depth in the middle asymptotically and gradually. For practical purposes, however, the depth may be considered constant if the variation in depth is within a certain margin, say, 1% of the average uniform-flow depth.

[2] In uniform flow, $S = S_f = S_w = S_0$. When the uniform-flow formula is applied to the computation of energy slope in a gradually varied flow, the energy slope will be denoted specifically by S_f instead of S.

A	the water area
V	the mean velocity
V_{ms}	the maximum surface velocity
P	the wetted perimeter
R	the hydraulic radius
y	the maximum depth of water area
S_w	the slope of the water surface
n	a coefficient representing the channel roughness, known as th *coefficient of roughness*[1]
Q_s	the suspended sediment charge
Q_b	the bed load
μ	the dynamic viscosity of the water
T	the temperature of the water

There have been developed and published a large number of practical uniform-flow formulas,[2] but none of these formulas meets the qualifications of a good formula as defined above. The best known and most widely used formulas are the Chézy and Manning formulas, which will be described in the following articles and used extensively in this book. Theoretical uniform-flow formulas have also been derived on the basis of a theoretical velocity distribution across the channel section, which will be discussed later (Art. 8-5).

A different approach to the determination of the velocity in a natural channel has been attempted by Toebes [6]. In this approach a multiple-correlation analysis is applied to the following significant factors affecting the velocity in a given alluvial channel: water area, maximum surface velocity, wetted perimeter, maximum depth, slope of water surface, coefficient of roughness, and temperature of water. By this method it is possible to evaluate the independent individual influence of each variable on the magnitude of the velocity. When such an evaluation is made, the velocity under any given condition of the variables is simply equal to the algebraic summation of the individual contributions as affected by each variable. However, this method applies only to the streams in the geographical region for which the analysis is made; hence, its application cannot be generalized.

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[1] In British literature the term "rugosity coefficient" is used.

[2] A number of well-known uniform-flow formulas are given and discussed in [2] to [5].

bution studied by Keulegan (Art. 8-4). The practical application of this formula is limited, since further investigation is needed for determination of the proper values of ϵ.

TABLE 5-3. TENTATIVE VALUES OF POWELL'S ϵ

Description of channel	Powell's ϵ	
	New	Old
Neat cement surface	0.0002	0.0004
Unplaned-plank flumes	0.0010	0.0017
Concrete-lined channels	0.004	0.006
Earth, straight and uniform	0.04	
Dredged earth channels	0.10	

Example 5-1. Compute the velocity and discharge in the trapezoidal channel described in Example 2-1, having a bottom width of 20 ft, side slopes 2:1, and a depth of water 6 ft. Given: Kutter's $n = 0.015$, and $S = 0.005$.

Solution. From Example 2-1, $A = 192.0$ ft² and $R = 4.10$ ft. Using the G. K. formula, the value of Chézy's C is

$$C = \frac{41.65 + \dfrac{0.00281}{0.005} + \dfrac{1.811}{0.015}}{1 + \left(41.65 + \dfrac{0.00281}{0.005}\right)\dfrac{0.015}{\sqrt{4.10}}} = 124.2$$

Then, by the Chézy formula,

$$V = 124.2\sqrt{4.10 \times 0.005} = 17.8 \text{ fps}$$

Therefore,

$$Q = 192.0 \times 17.8 = 3{,}420 \text{ cfs}$$

5-6. The Manning Formula. In 1889 the Irish engineer Robert Manning[1] presented a formula, which was later modified to its present

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[1] Manning first presented the formula in a paper read on December 4, 1889, at a meeting of the Institution of Civil Engineers of Ireland. The paper was later published in the Transactions of the Institution [15]. The formula was first given in a complicated form and then simplified to $V = CR^{2/3}S^{1/2}$, where V is the mean velocity, C is a factor of flow resistance, R is the hydraulic radius, and S is the slope. This was further modified by others and expressed in metric units as $V = (1/n)R^{2/3}S^{1/2}$. Later, it was converted back again to English units, resulting in $V = (1.486/n)R^{2/3}S^{1/2}$. In this conversion, as in the conversion of the Ganguillet and Kutter formula, the numerical value of n is kept unaffected. Consequently, the same value of n is widely used in both systems of units.

In the view of modern fluid mechanics, which pays much attention to dimensions, the dimensions of n become a matter of consideration. Directly from the Manning formula, the dimensions of n are seen to be $TL^{-1/3}$. Since it is unreasonable to suppose that the roughness coefficient would contain the dimension T, some authors assume that the numerator contains $\sqrt{g}$, thus yielding the dimensions of $L^{1/6}$ for n. Also, for physical reasons, it will be seen that $n = [\phi(R/k)]k^{1/6}$ [Eq. (8-26)], where k is well-known form

$$V = \frac{1.49}{n} R^{2/3} S^{1/2} \tag{5-6}$$

where V is the mean velocity in fps, R is the hydraulic radius in ft, S is the slope of energy line, and n is the coefficient of roughness, specifically known as *Manning's n*. This formula was developed from seven different formulas, based on Bazin's experimental data, and further verified by 170 observations.[1] Owing to its simplicity of form and to the satis-

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a linear measure of roughness and $\phi(R/k)$ is a function of R/k. If $\phi(R/k)$ is considered dimensionless, n will have the same dimensions as those of $k^{1/6}$, that is, $L^{1/6}$.

On the other hand, of course, it is equally possible to assume that the numerator of $1.486/n$ can absorb the dimensions of $L^{1/3}T^{-1}$, or that $\phi(R/k)$ involves a dimensional factor, thus leaving no dimensions for n. Some authors, therefore, preferring the simpler choice, consider n a dimensionless coefficient.

It is interesting to note that the conversion of the units for the Manning formula is independent of the dimensions of n, as long as the same value of n is used in both systems of units. If n is assumed dimensionless, then the formula in English units gives the numerical constant $3.2808^{1/3} = 1.486$ since 1 meter = 3.2808 ft. Now, if n is assumed to have the dimensions of $L^{1/6}$, its numerical value in English units must be different from its value in metric units, unless a numerical correction factor is introduced for compensation. Let n be the value in metric units and n' the value in English units. Then, $n' = (3.2808^{1/6})n = 1.2190n$. When the formula is converted from metric to English units, the resulting form takes the numerical constant $3.2808^{1/3+1/6} = 3.2808^{1/2} = 1.811$, since n has the dimensions of $L^{1/6}$. Thus, the resulting equation should be written $V = 1.811R^{2/3}S^{1/2}/n'$. Since the same value of n is used in both systems, the practical form of the formula in the English system is $V = 1.811R^{2/3}S^{1/2}/1.2190n = 1.486R^{2/3}S^{1/2}/n$, which is identical with the formula derived on the assumption that n has no dimensions.

In a search of early literature on hydraulics, the author has failed to find any significant discussion regarding the dimensions of n. It seems that this was not a problem of concern to the forefathers of hydraulics. It is most likely, however, that n was unconsciously taken as dimensionless in the conversion of the Manning formula, because such a conversion, as shown above, is more direct and simpler.

Now, considering the approximations involved in the derivation of the formula and the uncertainty in the value of n, it seems unjustifiable to carry the numerical constant to more than three significant figures. For practical purposes, a value of 1.49 is believed to be sufficiently accurate [16].

Manning mentioned that the simplified form of the formula had been suggested independently by G. H. L. Hagen prior to Manning's own work, according to a statement by Major Cunningham [17]. Hagen's formula was believed to have appeared first in 1876 [7]. It is also known that Philippe-Gaspard Gauckler [18] had an early proposal of the simplified form of Manning's formula in 1868 and that Strickler [19] presented independently the same form of the formula in 1923.

[1] For the derivation of the exponent of R, use was made of Bazin's experimental data on artificial channels [12]. For different shapes and roughnesses, the average value of the exponent was found to vary from 0.6499 to 0.8395. Considering these variations, Manning adopted an approximate value of $2/3$ for the exponent. On the

factory results it lends to practical applications, the Manning formula has become the most widely used of all uniform-flow formulas for open-channel flow computations.[1] A nomographic solution of the formula is given in Appendix C.

Within the normal ranges of slope and hydraulic radius, the values of Manning's n and Kutter's n are generally found to be numerically very close. For practical purposes, the two values may be considered identical when the slope is equal to or greater than 0.0001 and the hydraulic radius is between 1.0 and 30 ft. Typical values good for both Kutter's n and Manning's n are shown in Table 5-6 and illustrated in Fig. 5-5.

Comparing the Chézy formula with the Manning formula, it can be seen that

$$C = \frac{1.49}{n} R^{1/6} \tag{5-7}$$

This equation provides an important relationship[2] between Chézy's C and Manning's n.

The exponent of the hydraulic radius in the Manning formula is actually not a constant but varies in a range depending mainly on the channel shape and roughness (see a previous footnote). For this reason, some hydraulicians prefer to use the formula with a variable exponent. For example, the uniform-flow formula widely used in the U.S.S.R. is of this type; this is the *Pavlovskiĭ formula* [21], proposed in 1925.* This formula in *metric* units is

$$C = \frac{1}{n} R^{y} \tag{5-8}$$

where $$y = 2.5\sqrt{n} - 0.13 - 0.75\sqrt{R}(\sqrt{n} - 0.10) \tag{5-9}$$

and where C is the resistance factor in the Chézy formula expressed in *metric* units. The exponent y depends on the roughness coefficient and hydraulic radius. The formula is valid for R between 0.1 and 3.0 m and

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basis of other later studies, some authors suggested a value of ¾ [20], and others suggested a variable depending on R and n [21].

[1] The Manning formula was suggested for international use by Lindquist [3] at the Scandinavia Sectional Meeting of the World Power Conference in 1933 in Stockholm. The final recommendation for such use was made by the Executive Committee at the 3d World Power Conference in 1936 in Washington, D.C.

[2] On account of this relationship, the Manning formula is sometimes considered a variation of the Chézy formula with Chézy's C defined by Eq. (5-7).

* The Pavlovskiĭ formula was published in several editions of Pavlovskiĭ's "Handbook of Hydraulics" [21]. An article about this formula entitled *Formula dlia koeffitsienta Chézy* (*Formula for a Chézy coefficient*) is given in pp. 140–149 of the 1937 edition of the book. A footnote in this article reads: "The formula was proposed in 1925."

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for n between 0.011 and 0.040. For practical purposes, the following approximate forms of Eq. (5-9) are generally suggested for use:

$$y = 1.5\sqrt{n} \quad \text{for } R < 1.0 \text{ m} \tag{5-10}$$

$$y = 1.3\sqrt{n} \quad \text{for } R > 1.0 \text{ m} \tag{5-11}$$

5-7. Determination of Manning's Roughness Coefficient. In applying the Manning formula or the G. K. formula, the greatest difficulty lies in the determination of the roughness coefficient n; for there is no exact method of selecting the n value. At the present stage of knowledge, to select a value of n actually means to estimate the resistance to flow in a given channel, which is really a matter of intangibles. To veteran engineers, this means the exercise of sound engineering judgment and experience; for beginners, it can be no more than a guess, and different individuals will obtain different results.

In order to give guidance in the proper determination of the roughness coefficient, four general approaches will be discussed; namely, (1) to understand the factors that affect the value of n and thus to acquire a basic knowledge of the problem and narrow the wide range of guesswork, (2) to consult a table of typical n values for channels of various types, (3) to examine and become acquainted with the appearance of some typical channels whose roughness coefficients are known, and (4) to determine the value of n by an analytical procedure based on the theoretical velocity distribution in the channel cross section and on the data of either velocity or roughness measurement. The first three approaches will be given in the next three articles, and the fourth approach will be taken up in Art. 8-7.

5-8. Factors Affecting Manning's Roughness Coefficient. It is not uncommon for engineers to think of a channel as having a single value of n for all occasions. In reality, the value of n is highly variable and depends on a number of factors. In selecting a proper value of n for various design conditions, a basic knowledge of these factors should be found very useful. The factors that exert the greatest influence upon the coefficient of roughness in both artificial and natural channels are therefore described below. It should be noted that these factors are to a certain extent interdependent; hence discussion about one factor may be repeated in connection with another.

A. Surface Roughness. The surface roughness is represented by the size and shape of the grains of the material forming the wetted perimeter and producing a retarding effect on the flow. This is often considered the only factor in selecting a roughness coefficient, but it is actually just one of several major factors. Generally speaking, fine grains result in a relatively low value of n and coarse grains, in a high value of n.

In alluvial streams where the material is fine in grain, such as sand,

CHAPTER 6

COMPUTATION OF UNIFORM FLOW

6-1. The Conveyance of a Channel Section. The discharge of uniform flow in a channel may be expressed as the product of the velocity, represented by Eq. (5-1), and the water area, or

$$Q = VA = CAR^xS^y = KS^y \tag{6-1}$$

where

$$K = CAR^x \tag{6-2}$$

The term K is known as the *conveyance* of the channel section; it is a measure of the carrying capacity of the channel section, since it is directly proportional to Q.

When either the Chézy formula or the Manning formula is used as the uniform-flow formula, i.e., when $y = \frac{1}{2}$, the discharge by Eq. (6-1) becomes

$$Q = K\sqrt{S} \tag{6-3}$$

and the conveyance is

$$K = \frac{Q}{\sqrt{S}} \tag{6-4}$$

This equation can be used to compute the conveyance when the discharge and slope of the channel are given.

When the Chézy formula is used, Eq. (6-2) becomes

$$K = CAR^{1/2} \tag{6-5}$$

where C is Chézy's resistance factor. Similarly, when the Manning formula is used,

$$K = \frac{1.49}{n} AR^{2/3} \tag{6-6}$$

The above two equations are used to compute the conveyance when the geometry of the water area and the resistance factor or roughness coefficient are given. Since the Manning formula is used extensively, most of the following discussions and computations will be based on Eq. (6-6).

6-2. The Section Factor for Uniform-flow Computation. The expression $AR^{2/3}$ is called the *section factor for uniform-flow computation;* it is an important element in the computation of uniform flow. From Eq.

(6-6), this factor may be expressed as

$$AR^{2/3} = \frac{nK}{1.49} \tag{6-7}$$

and, from Eq. (6-4),

$$AR^{2/3} = \frac{nQ}{1.49\sqrt{S}} \tag{6-8}$$

Primarily, Eq. (6-8) applies to a channel section when the flow is uniform. The right side of the equation contains the values of n, Q, and S; but the left side depends only on the geometry of the water area. Therefore, it shows that, for a given condition of n, Q, and S, there is *only one* possible depth for maintaining a uniform flow, provided that the value of $AR^{2/3}$ always increases with increase in depth, which is true in most cases. This depth is the *normal depth.* When n and S are known at a channel section, it can be seen from Eq. (6-8) that there can be *only one* discharge for maintaining a uniform flow through the section, provided that $AR^{2/3}$ always increases with increase of depth.[1] This discharge is the *normal discharge.*

Equation (6-8) is a very useful tool for the computation and analysis of uniform flow. When the discharge, slope, and roughness are known, this equation gives the section factor $A_nR_n^{2/3}$ and hence the normal depth y_n. On the other hand, when n, S, and the depth, hence the section factor, are given, the normal discharge Q_n can be computed from this equation in the following form:

$$Q = \frac{1.49}{n} AR^{2/3}\sqrt{S} \tag{6-9}$$

This is essentially the product of the water area and the velocity defined by the Manning formula. The subscript n is sometimes used to specify the condition of uniform flow.

In order to simplify the computation, dimensionless curves showing the relation between depth and section factor $AR^{2/3}$ (Fig. 6-1) have been prepared for rectangular, trapezoidal, and circular channel sections. These self-explanatory curves will help to determine the depth for a given section factor $AR^{2/3}$, and vice versa. The $AR^{2/3}$ values for a circular section can also be found from the table in Appendix A.

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[1] This is true for channels in which the value of $AR^{2/3}$ always increases with increase of depth, since Eq. (6-8) will give one value of $AR^{2/3}$, which in turn gives only one depth. In the case of a closed conduit having a gradually closing top, the value of $AR^{2/3}$ will first increase with depth and then decrease with depth when the full depth is approached, because a maximum value of $AR^{2/3}$ usually occurs in such a conduit at a depth slightly less than the full depth. Consequently, it is possible to have two depths for the same value of $AR^{2/3}$, one greater and the other less than the depth for the maximum value of $AR^{2/3}$. For further discussion on this subject see Art. 6-4.

CHAPTER 9

THEORY AND ANALYSIS

9-1. Basic Assumptions. The gradually varied flow to be discussed in Part III of this book is the steady flow whose depth varies gradually along the length of the channel (Art. 1-2). This definition signifies two conditions: (1) that the flow is steady; that is, that the hydraulic characteristics of flow remain constant for the time interval under consideration; and (2) that the streamlines are practically parallel; that is, that hydrostatic distribution of pressure prevails over the channel section.

The development of the theory of gradually varied flow dates back to the eighteenth century. Many early hydraulicians[1] have contributed to this development. The theories thus developed practically all hinge on the following basic assumption:

A. The head loss at a section is the same as for a uniform flow having the velocity and hydraulic radius of the section.

According to this assumption, *the uniform-flow formula may be used to evaluate the energy slope of a gradually varied flow at a given channel section, and the corresponding coefficient of roughness developed primarily for uniform flow is applicable to the varied flow.* This assumption has never been precisely confirmed by either experiment[2] or theory, but errors arising from it are believed to be small compared with those ordinarily incurred in the use of a uniform-flow formula and in the selection of the roughness coefficient. Over years of use this assumption has proved to be a reliable basis for design. The assumption is undoubtedly more correct for varied flow where the velocity increases than where the velocity decreases, because in a flow of increasing velocity the head loss is caused almost entirely by frictional effects, whereas in a flow of decreasing velocity there may be large-scale eddy losses.

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[1] Bélanger [1] is believed to be the outstanding contributor. Also among early contributors are Bernoulli, Bresse, Poncelet, Saint-Venant, Boussinesq, and others [2] to [5].

[2] Using the experimental data from the Sunderland Technical College and King's College in England and from the University of Illinois, Bettes [6] has derived an f-R relationship (Art. 1-3) for gradually varied flow in smooth open channels, which was found to agree very closely with the relationship for uniform flow obtained by Allen [7]. Also, the computation of backwater curves based on this assumption has been verified satisfactorily by many experiments. These experimental verifications, though not very rigorous, indicate the validity of the assumption for practical purposes.

In addition to the above basic assumption, the following assumptions will also be used where further simplification is necessary in subsequent discussions:

B. The slope of the channel is small; so that:

1. The depth of flow is the same whether the vertical or normal (to the channel bottom) direction is used.

2. The pressure-correction factor $\cos\theta$ [applied to the depth of the flow section, Eq. (2-12)] is equal to unity.

3. No air entrainment occurs. In case of notable air entrainment, the computation may be carried out assuming no entrainment and then corrected approximately, at the end, using Eq. (2-15).

C. The channel is prismatic; that is, the channel has constant alignment and shape.

D. The velocity distribution in the channel section is fixed. Thus, the velocity-distribution coefficients are constant.

E. The conveyance K (Art. 6-3) and section factor Z (Art. 4-3) are exponential functions of the depth of flow.

F. The roughness coefficient is independent of the depth of flow and constant throughout the channel reach under consideration.

9-2. Dynamic Equation of Gradually Varied Flow. Consider the profile of gradually varied flow in the elementary length dx of an open channel (Fig. 9-1). The total head above the datum at the upstream section 1 is

$$H = z + d\cos\theta + \alpha\frac{V^2}{2g} \tag{3-2}$$

where H is the total head in ft; z is the vertical distance of the channel bottom above the datum in ft; d is the depth of flow section in ft; θ is the bottom-slope angle; α is the energy coefficient; and V is the mean velocity of flow through the section in fps.

It is assumed that θ and α are constant throughout the channel reach under consideration. Taking the bottom of the channel as the x axis and differentiating Eq. (3-2) with respect to the length x of the water-surface profile, which is measured along the x axis, the following equation is obtained:

$$\frac{dH}{dx} = \frac{dz}{dx} + \cos\theta\frac{dd}{dx} + \alpha\frac{d}{dx}\left(\frac{V^2}{2g}\right) \tag{9-1}$$

It should be noted that the slope is defined as the sine of the slope angle and that it is assumed *positive* if it *descends* in the direction of flow and *negative* if it *ascends*. Hence,[1] in Fig. 9-1, the energy slope $S_f = -dH/dx$,

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[1] It should be noted that the frictional loss dH is always a negative quantity in the direction of flow (unless outside energy is added to the course of the flow) and that the change in the bottom elevation dz is a negative quantity when the slope descends.

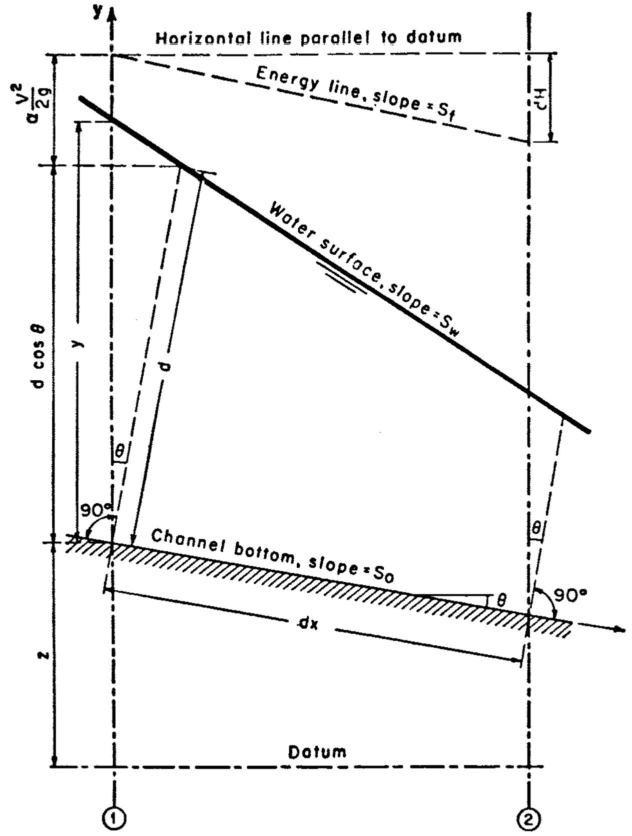
y
Horizontal line parallel to datum
Energy line, slope = S_f
$\alpha \frac{V^2}{2g}$
dH
Water surface, slope = S_w
d cos θ
y
d
θ
90°
Channel bottom, slope = S_0
θ
θ
90°
dx
z
Datum
①
②

FIG. 9-1. Derivation of the gradually-varied-flow equation.

and the slope of the channel bottom $S_0 = \sin\theta = -dz/dx$. Substituting these slopes in Eq. (9-1) and solving for dd/dx,

$$\frac{dd}{dx} = \frac{S_0 - S_f}{\cos\theta + \alpha\, d(V^2/2g)/dd} \tag{9-2}$$

This is the general differential equation for gradually varied flow, referred to hereafter as the *dynamic equation of gradually varied flow*, or simply as the *gradually-varied-flow equation.* It represents the slope of the water surface *with respect to the bottom of the channel.* The depth d is measured from the bottom of the channel, and the channel bottom is taken as the x axis. Thus, the slope of the water surface is equal to the bottom slope S_0 if $dd/dx = 0$, less than S_0 if dd/dx is positive, and greater than S_0 if dd/dx is negative. In other words, the water surface is parallel to where Q is the given discharge of the gradually varied flow at the actual depth y; Q_n is the normal discharge at a depth equal to y; and Q_c is the critical discharge at a depth equal to y; and

$$\frac{dy}{dx} = \frac{S_0 - Q^2/C^2A^2R}{1 - \alpha Q^2/gA^2D} \tag{9-16}$$

where D is the hydraulic depth, C is Chézy's resistance factor, and the rest of the notation is as defined in this article.

For wide rectangular channels,

1. When the Manning formula is used,

$$\frac{dy}{dx} = S_0 \frac{1 - (y_n/y)^{10/3}}{1 - (y_c/y)^3} \tag{9-17}$$

2. When the Chézy formula is used,

$$\frac{dy}{dx} = S_0 \frac{1 - (y_n/y)^3}{1 - (y_c/y)^3} \tag{9-18}$$

9-3. Characteristics of Flow Profiles. The dynamic equation of gradually varied flow developed in Art. 9-2 expresses the longitudinal surface slope of the flow with respect to the channel bottom. It can therefore be used to describe the characteristics of various *flow profiles* or profiles of the water surface of the flow. For simplicity, the channel is considered *prismatic*, and Eq. (9-13) is used for discussion. The values of K and Z in this equation are assumed to increase or decrease continuously with the depth y. This is true for all open-channel sections except for conduits with a gradually closing top. In such conduits, the value of K, after reaching its maximum value, will decrease as the depth of flow approaches the top of the conduit (Art. 6-3).

The flow profile represents the *surface curve* of the flow. It will represent a backwater curve[1] (Art. 4-5) if the depth of flow increases in the direction of flow and a drawdown curve (Art. 4-5) if the depth decreases in the direction of flow. Following the description in the preceding article, it can be seen that the flow profile is a backwater curve if dy/dx is positive and a drawdown curve if dy/dx is negative.

For a backwater curve, dy/dx is positive; thus, Eq. (9-13) gives two possible cases:

1. $1 - (K_n/K)^2 > 0$ and $1 - (Z_c/Z)^2 > 0$
2. $1 - (K_n/K)^2 < 0$ and $1 - (Z_c/Z)^2 < 0$

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[1] The term "backwater curve" is used primarily to indicate the longitudinal surface curve of the water backed up above a dam or into a tributary by flood in the main stream. Many authors have extended its meaning to include all types of flow profiles.

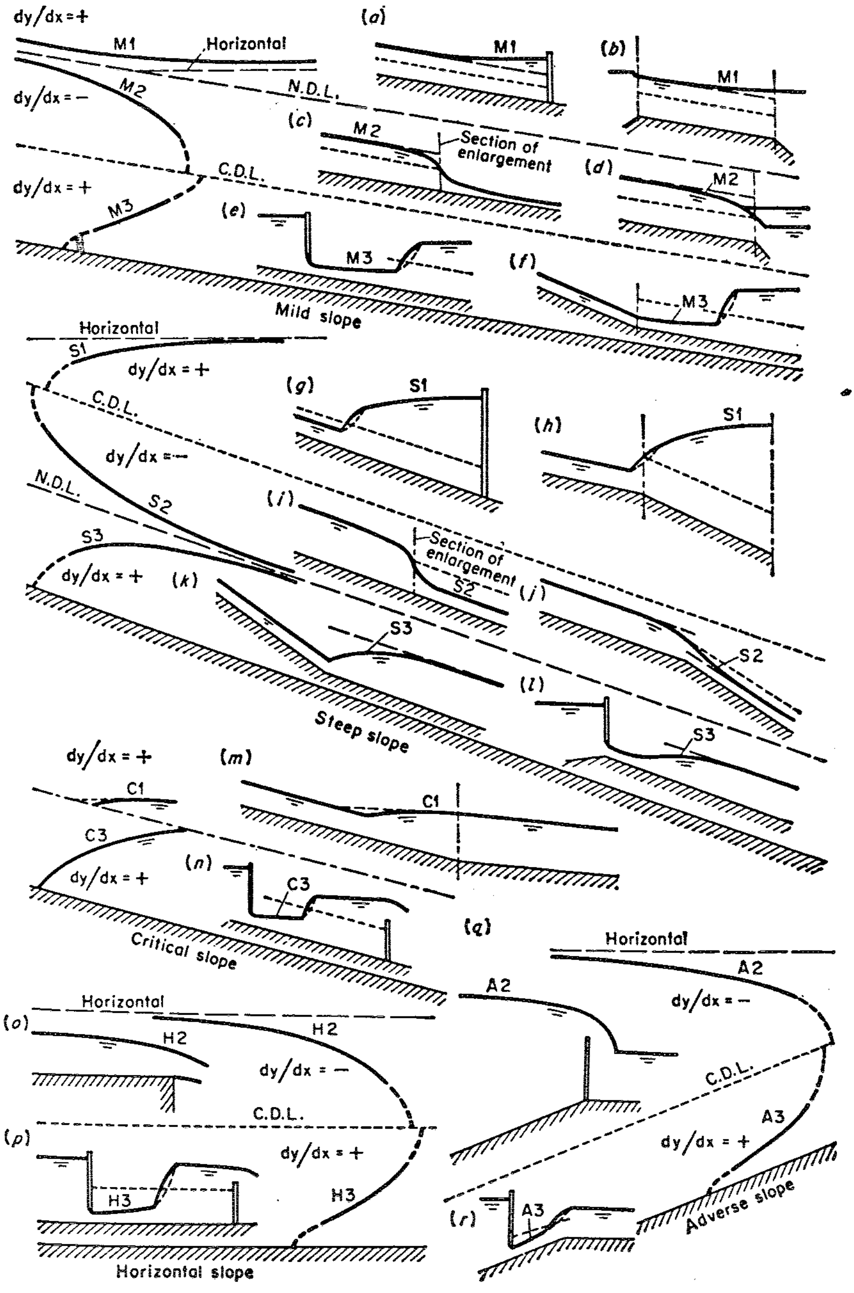

FIG. 9-4. Examples of flow profiles.

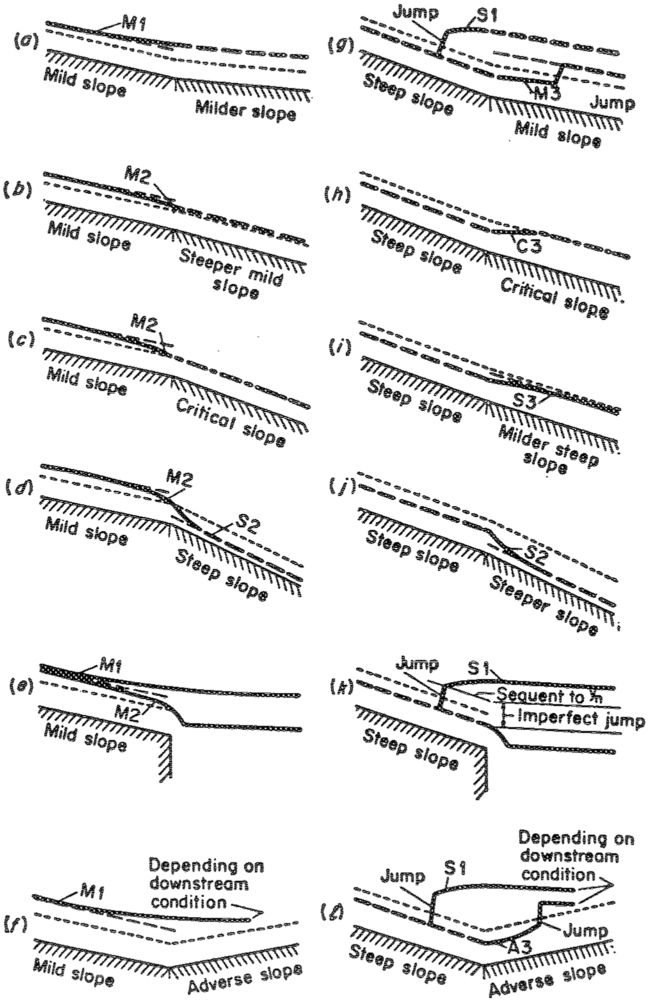

FIG. 9-6. Profiles of gradually varied flow in a long prismatic channel with a break in bottom slope.

9-5. Analysis of Flow Profile. Flow-profile analysis is a procedure used to predict the general shape of the flow profile. It enables the engineer to learn beforehand the possible flow profiles that may occur in a given channel layout. This procedure constitutes a very significant part of all problems in channel design for gradually varied flow.

A. Prismatic Channel with Constant Slope. The flow profile in a long prismatic channel with a constant slope has been described in Art. 9-4.

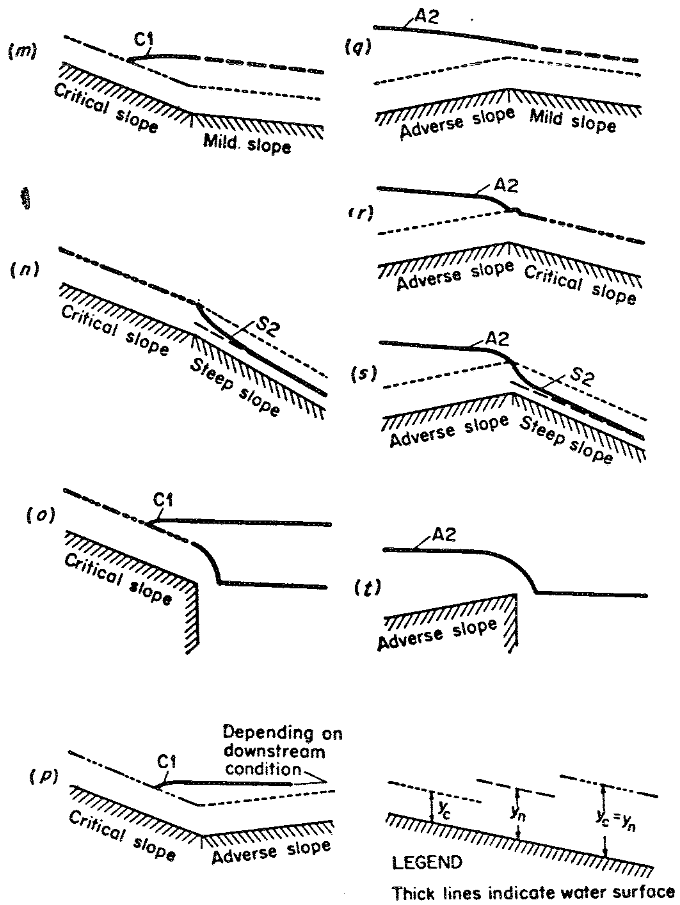

FIG. 9-6 (*Continued*).

Typical examples shown in Fig. 9-4 should be helpful in determining the type of flow profile in a given problem.

B. Prismatic Channel with a Change in Slope. This channel is equivalent to a pair of connected prismatic channels of the same cross section but with different slopes. Twenty typical flow profiles in a long prismatic channel with a break in slope are shown in Fig. 9-6. These profiles are self-explanatory. However, some special features should be mentioned:

1. The profile near or at the critical depth cannot be predicted precisely by the theory of gradually varied flow, since the flow is generally rapidly varied.
2. In passing a critical line, the flow profile should, theoretically, have a

vertical slope. Since the flow is usually rapidly varied when passing the critical line, the actual slope of the profile cannot be predicted precisely by the theory. For the same reason, the critical depth may not occur exactly above the break of the channel bottom and may be different from the depth shown in the figure.

3. In some cases (Fig. 9-6*g* and *l*), the hydraulic jump may occur either in the upstream channel or in the downstream channel, depending upon the relative steepness of the two slopes.[1] In case *g*, for instance, the jump will occur in the downstream channel if the normal depth in this channel is comparatively small. When the slope of the downstream channel decreases and, accordingly, the normal depth increases, the jump will move upstream, eventually into the upstream channel. The exact location of the jump will be discussed in Art. 15-7.

4. If the upstream channel has an adverse slope (Fig. 9-6*q* to *t*), the discharge is fixed not by upstream channel conditions but by the elevation of the upstream pool level, which is the horizontal asymptote of the *A*2 profile. The procedure of analysis is to assume a discharge and to determine to which case *q* to *t* the profile should belong. Then, compute the flow profile in the upstream direction and determine the pool level. If the computed level does not agree with the given pool level, then repeat the computation with another assumed discharge until the computed level agrees with the given level.

5. Typical profiles (Fig. 9-6) are illustrated for long channels in which a uniform flow can be established far upstream and downstream.

C. Prismatic Channel with Several Changes in Slope. For such channels the general procedure of analysis is as follows:

1. Plot the channel profile with an exaggerated vertical scale.

2. Compute y_n for each reach, and plot the normal-depth line, shown by dashed lines, throughout the entire channel.

3. Compute y_c for each reach, and plot the critical-depth line, shown by dotted lines, throughout the channel.

4. Locate all possible control sections. At the control section,[2] flow must pass through a control depth which may be the critical depth, the normal depth, or any other known depth. There are three types of control section:

a. UPSTREAM CONTROL SECTION. This occurs in any steep reach at the upstream end, since the flow in a steep channel has to pass through the critical section at the upstream end and then follow either the *S*1 or *S*2 profile. The critical depth is, therefore, the control depth (see also Art.

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[1] Also depending on the relative roughness and shape of the two connecting channels. In this discussion these factors are assumed constant.

[2] The term "control section" used here has a broad meaning. It refers to any section at which the depth of flow is known or can be controlled to a required stage.

4-5). If the downstream water surface is very high, it may raise the flow surface at the upstream control. When several steep reaches occur in succession, the control section is at the upstream end of the uppermost reach. Upstream control also occurs in long mild reaches, because the $M1$ or $M2$ curves will approach the normal depth at the upstream end.

b. DOWNSTREAM CONTROL SECTION. This occurs at the downstream end in any long steep reach, because the flow will approach the normal depth at the downstream end. If the downstream end of a mild channel terminates at a free overfall, the control section may be assumed at the brink where the depth is critical.[1]

c. ARTIFICIAL CONTROL SECTION. This occurs at a control structure, such as a weir, dam, or sluice gate, at which the control depth either is known or can be determined.

5. Starting at the control depth at each control section, trace in each reach a continuous profile. The position of the profile in each reach can be correctly located with respect to the normal- and critical-depth lines. For this purpose, typical profiles described previously (Art. 9-4) should be found useful.

6. When flow is supercritical in the upstream portion of a reach but subcritical in the downstream portion, the flow profile has to pass the critical depth somewhere in the reach. In crossing the critical-depth line, a hydraulic jump is usually created in raising the water surface from a low depth to its sequent depth. The exact location of the jump will be discussed later (Art. 15-7).

Example 9-1. The normal and critical depths of the flow in a channel have been computed and are shown in Fig. 9-7. Sketch the possible flow profiles.

Solution. First locate the possible control sections, such as those for upstream control U.C., downstream control D.C., and artificial control A.C. The artificial control in this example is a sluice which backs up water to form an $S1$ profile on the upstream side. The formation of the hydraulic jumps in the middle and downstream reaches is apparent, but the determination of their exact positions requires further consideration. In the lowermost reach, for instance, the $M2$ and $M3$ profiles should be computed first. Then, compute the curve representing the sequent depth of the $M3$ profile. The intersection of the sequent-depth curve and the $M2$ profile gives the approximate position of the jump. A more exact value for the position of the jump may be obtained by correcting the position for the length of the jump. Various types of flow profile are sketched in the figure.

D. Nonprismatic Channels and Channels with Spatially Varied Flow. In nonprismatic channels and channels with spatially varied flow, the analysis of flow becomes complicated by the fact that the control section may occur at any section in the channel and its position cannot be determined easily. Consider three different channel slopes for three types of flow: (1) continuous flow in a prismatic channel (Fig. 9-8*a*), (2) spatially

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[1] Actually the computed critical depth is somewhat behind the brink (Art. 3-4).

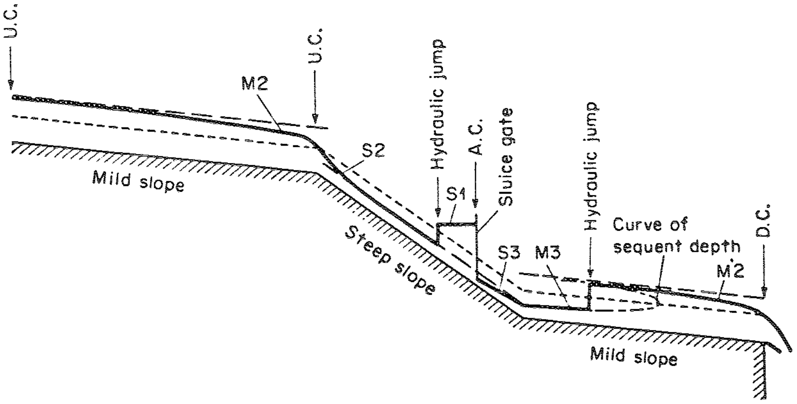

Fig. 9-7. Analysis of flow profile for Example 9-1. U.C. = upstream control; D.C. = downstream control; A.C. = artificial control.

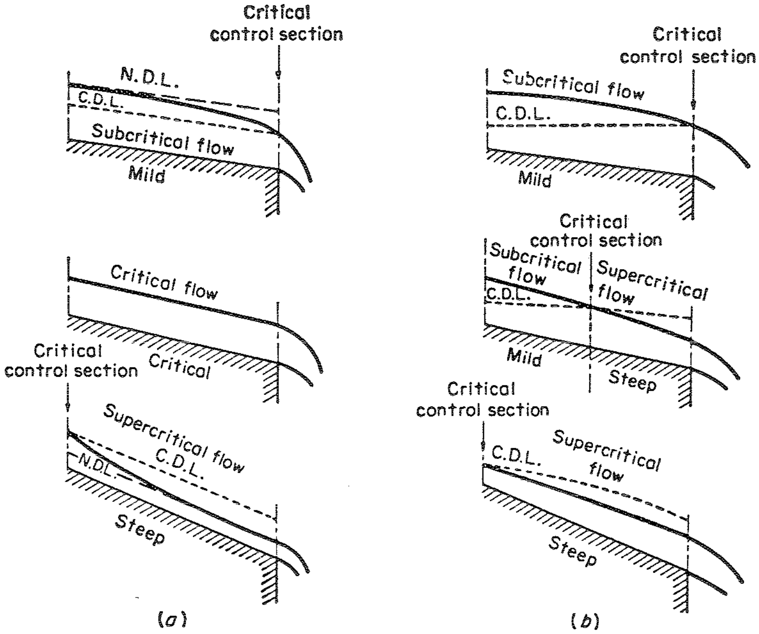

Fig. 9-8. Location of critical control section. (*a*) Flow in a prismatic channel; (*b*) flow in a nonprismatic channel or spatially varied flow.

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9-6. M
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varied flow of increasing discharge in a prismatic channel (Fig. 9-8*b*), and (3) continuous flow in a nonprismatic channel having a constant slope but varying cross section (Fig. 9-8*b*). All channels have a free overfall.

In the first type of flow, the flow changes from subcritical to supercritical as the channel slope varies from mild to steep. Accordingly, the critical control section will be transferred from the downstream end to the upstream end. It should be noted that the transfer of the critical control section occurs almost instantaneously. At the instant of transition,[1] the critical flow takes place throughout the entire length of the channel (middle view of Fig. 9-8*a*), and any section in the reach is, therefore, a critical section.

In the second and third types of flow, the transition of the state of flow will take place gradually from section to section as the critical control section moves upstream. At any moment during the transition (middle view of Fig. 9-8*b*), the flow downstream from the control section is supercritical, and the flow upstream is subcritical. The determination of the control section will be described in the next article.

9-6. Method of Singular Points. The preceding analysis of flow profiles was given primarily for prismatic channels. For a comprehensive treatment of flow profiles, in either prismatic or nonprismatic channels with constant or variable slope, use has been made of advanced mathematical approaches.[2] One approach is the use of the theory of the singular point. This theory was developed by Poincaré [22] but was first applied to flow studies in channels of variable slope by Massé [23]. The method based on this theory has been further discussed and extended by Jaeger [24], De Marchi [25], Hom-ma [26,27], Escoffier [28], Iwasa [29], and others.

For simplicity of discussion, channels with small slopes will be considered. Let the numerator and denominator of Eq. (9-15) be represented by two functions, or

$$\frac{dy}{dx} = S_0 \frac{1 - (Q/Q_n)^2}{1 - (Q/Q_c)^2} = \frac{F_1(x,y)}{F_2(x,y)} \quad \text{or} \quad \frac{F_1}{F_2} \tag{9-20}$$

Then, set each of these functions equal to zero, or

$$F_1 = S_0 \left[1 - \left(\frac{Q}{Q_n} \right)^2 \right] = 0 \tag{9-21}$$

and

$$F_2 = 1 - \left(\frac{Q}{Q_c} \right)^2 = 0 \tag{9-22}$$

The solution of Eq. (9-21) $F_1 = 0$ will give $Q = Q_n$ or $y = y_n$. Hence,

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[1] At this moment, the normal depth coincides with the critical depth, becoming a so-called transitional depth (Art. 9-6).

[2] See another method of analysis by Merten [20,21].

CHAPTER 10

METHODS OF COMPUTATION

The computation of gradually-varied-flow profiles involves basically the solution of the dynamic equation of gradually varied flow. The main objective of the computation is to determine the shape of the flow profile. Broadly classified, there are three methods of computation; namely, the graphical-integration method, the direct-integration method, and the step method. The development and procedure of several typical methods will be described in this chapter.

10-1. The Graphical-integration Method. This method is to integrate the dynamic equation of gradually varied flow by a graphical procedure. Consider two channel sections (Fig. 10-1a) at distances x_1 and x_2, respectively, from a chosen origin and with corresponding depths of flow y_1 and y_2. The distance along the channel floor is

$$x = x_2 - x_1 = \int_{x_1}^{x_2} dx = \int_{y_1}^{y_2} \frac{dx}{dy}\, dy \tag{10-1}$$

Assume several values of y, and compute the corresponding values of dx/dy, which is the reciprocal of the right-side member of a gradually-varied-flow equation, say Eq. (9-13). A curve of y against dx/dy is then constructed (Fig. 10-1b). According to Eq. (10-1), it is apparent that the value of x is equal to the shaded area formed by the curve, the y axis, and the ordinates of dx/dy corresponding to y_1 and y_2. This area can be measured and the value of x determined.

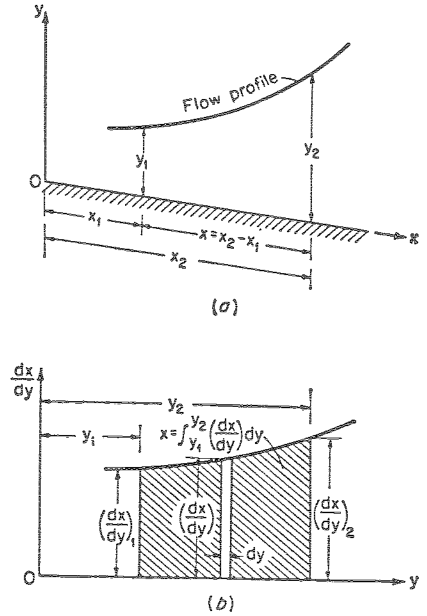
y
Flow profile
y_1
y_2
O
x_1
$x = x_2 - x_1$
x_2
x
(a)
$\frac{dx}{dy}$
y_2
y_1
$x = \int_{y_1}^{y_2} \left(\frac{dx}{dy}\right) dy$
$\left(\frac{dx}{dy}\right)_1$
$\left(\frac{dx}{dy}\right)$
dy
$\left(\frac{dx}{dy}\right)_2$
O
y
(b)

FIG. 10-1. Principle of the graphical-integration method.

where $(K_0/K)^2$ is evidently a function of y/d_0 and, hence, can be represented by $f_1(y/d_0)$.

From Eqs. (9-4) and (9-7), the following may be written:

$$\left(\frac{Z_c}{Z}\right)^2 = \frac{\alpha Q^2 T}{gA^3} = \frac{\alpha Q^2}{d_0^5}\frac{T/d_0}{g(A/d_0^2)^3} = \frac{\alpha Q^2}{d_0^5} f_2\left(\frac{y}{d_0}\right) \tag{10-33}$$

where $(T/d_0)/g(A/d_0^2)^3$ is apparently a function of y/d_0 and, hence, can be represented by $f_2(y/d_0)$.

Substituting Eqs. (10-32) and (10-33) in Eq. (9-13) and simplifying,

$$dx = \frac{d_0}{S_0}\left[\frac{1 - (\alpha Q^2/d_0^5)f_2(y/d_0)}{1 - (Q/Q_0)^2 f_1(y/d_0)}\right] d\left(\frac{y}{d_0}\right) \tag{10-34}$$

Integrating,

$$x = \frac{d_0}{S_0}\left[\int_0^{y/d_0} \frac{d(y/d_0)}{1 - (Q/Q_0)^2 f_1(y/d_0)} - \frac{\alpha Q^2}{d_0^5}\int_0^{y/d_0} \frac{f_2(y/d_0)\, d(y/d_0)}{1 - (Q/Q_0)^2 f_1(y/d_0)}\right] + \text{const} \tag{10-35}$$

or

$$x = -\frac{d_0}{S_0}\left(X - \frac{\alpha Q^2}{d_0^5}\, Y\right) + \text{const} \tag{10-36}$$

where

$$X = F_1\left(\frac{y}{d_0}, \frac{Q}{Q_0}\right) = \int_0^{y/d_0} \frac{-d(y/d_0)}{1 - (Q/Q_0)^2 f_1(y/d_0)} \tag{10-37}$$

and

$$Y = F_2\left(\frac{y}{d_0}, \frac{Q}{Q_0}\right) = \int_0^{y/d_0} \frac{-f_2(y/d_0)\, d(y/d_0)}{1 - (Q/Q_0)^2 f_1(y/d_0)} \tag{10-38}$$

These are the varied-flow functions for circular conduits, depending on y/d_0 and Q/Q_0. They can be evaluated by a procedure of numerical integration, say Simpson's rule. A table of these functions for positive slopes,[1] prepared by Keifer and Chu, is given in Appendix E.

The length of flow profile between two consecutive sections of depth y_1 and y_2, respectively, in a circular conduit may be expressed as

$$L = A[(X_2 - X_1) - B(Y_2 - Y_1)] \tag{10-39}$$

where $A = -d_0/S_0$ and $B = \alpha Q^2/d_0^5$.

10-3. The Direct Step Method. In general, a step method is characterized by dividing the channel into short reaches and carrying the computation step by step from one end of the reach to the other. There is a great variety of step methods. Some methods appear superior to others in certain respects, but no one method has been found to be the best in all applications. The direct step method[1] is a simple step method applicable to prismatic channels.

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[1] If $S_0 = 0$, then $Q_0 = 0$, $Q/Q_0 = \infty$, and the varied-flow functions become meaningless. If S_0 is negative, Eq. (10-30) shows that Q_0^2 is negative. Since the actual discharge Q must be positive, $(Q/Q_0)^2$ becomes negative. Thus, the integration procedure must be done for negative values of $(Q/Q_0)^2$ in the two varied-flow functions.

Figure 10-6 illustrates a short channel reach of length Δx. Equating the total heads at the two end sections 1 and 2, the following may be written:

$$S_0 \Delta x + y_1 + \alpha_1 \frac{V_1^2}{2g} = y_2 + \alpha_2 \frac{V_2^2}{2g} + S_f \Delta x \qquad (10\text{-}40)$$

Solving for Δx,

$$\Delta x = \frac{E_2 - E_1}{S_0 - S_f} = \frac{\Delta E}{S_0 - S_f} \qquad (10\text{-}41)$$

where E is the specific energy or, assuming $\alpha_1 = \alpha_2 = \alpha$,

$$E = y + \alpha \frac{V^2}{2g} \qquad (10\text{-}42)$$

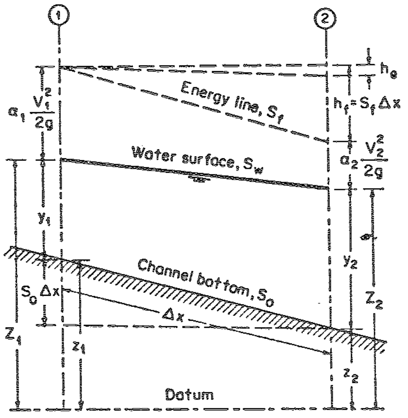

FIG. 10-6. A channel reach for the derivation of step methods.

In the above equations, y is the depth of flow, V is the mean velocity, α is the energy coefficient, S_0 is the bottom slope, and S_f is the friction slope. The average value of S_f is denoted by $\bar{S}_f$. When the Manning formula is used, the friction slope is expressed by

$$S_f = \frac{n^2 V^2}{2.22 R^{4/3}} \qquad (9\text{-}8)$$

The direct step method is based on Eq. (10-41), as may be illustrated by the following example:

Example 10-7. Compute the flow profile required in Example 10-1 by the direct step method.

Solution. With the data given in Example 10-1, the step computations are carried out as shown in Table 10-4. The values in each column of the table are explained as follows:

Col. 1. Depth of flow in ft, arbitrarily assigned from 5.00 to 3.40 ft
Col. 2. Water area in ft² corresponding to the depth y in col. 1
Col. 3. Hydraulic radius in ft corresponding to y in col. 1
Col. 4. Four-thirds power of the hydraulic radius
Col. 5. Mean velocity in fps obtained by dividing 400 cfs by the water area in col. 2
Col. 6. Velocity head in ft
Col. 7. Specific energy in ft obtained by adding the velocity head in col. 6 to the depth of flow in col. 1

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[1] First suggested by the Polish engineer Charnomskii [22] in 1914 and then by Husted [23] in 1924.

Col. 8. Change of specific energy in ft, equal to the difference between the E value in col. 7 and that of the previous step

Col. 9. Friction slope computed by Eq. (9-8) with $n = 0.025$ and with V as given in col. 5 and $R^{4/3}$ in col. 4

Col. 10. Average friction slope between the steps, equal to the arithmetic mean of the friction slope just computed in col. 9 and that of the previous step

Col. 11. Difference between the bottom slope 0.0016 and the average friction slope

Col. 12. Length of the reach in ft between the consecutive steps, computed by Eq. (10-41) or by dividing the value of ΔE in col. 8 by the value in col. 11

Col. 13. Distance from the section under consideration to the dam site. This is equal to the cumulative sum of the values in col. 12 computed for previous steps.

The flow profile thus computed is practically identical with that obtained by graphical integration (Fig. 10-3).

Example 10-8. A 72-in. reinforced-concrete pipe culvert, 250 ft long, is laid on a slope of 0.02 with a free outlet. Compute the flow profile if the culvert discharges 252 cfs, $n = 0.012$, and $\alpha = 1.0$.

Solution. From the data, $y_c = 4.35$ ft and $y_n = 2.60$ ft. Since $y_c > y_n$, the channel slope is steep. As shown in Fig. 10-7, the control section is at the entrance; water will enter the culvert at the critical depth and thereafter flow at a depth less than y_c but greater than y_n. The flow profile is of the $S2$ type.

Table 10-5 shows the computation of the flow profile, which is self-explanatory. The computed profile is plotted as shown in Fig. 10-7. Plotted also in the figure is the energy line indicating the variation of energy along the culvert. The computation has been carried to exceed the length of the culvert, so that the depth of flow at the outlet can be interpolated. This depth is found to be 2.81 ft, and the corresponding outlet velocity is 19.4 fps. It should be noted that, if the pipe were flowing full at the outlet, the outlet velocity would be only 10 fps.

Note that, in either the direct step method or the standard step method which will be described in the next article, the step computation should be carried upstream if the flow is subcritical and downstream if the flow is supercritical. Step computations carried in the wrong direction tend inevitably to make the result diverge from the correct flow profile.

10-4. The Standard Step Method. This method is applicable also to nonprismatic channels. In nonprismatic channels, the hydraulic elements are no longer independent of the distance along the channel. In natural channels, it is generally necessary to conduct a field survey to collect the data required at all sections considered in the computation. The computation is carried on by steps from station to station where the hydraulic characteristics have been determined. In such cases the distance between stations is given, and the procedure is to determine the depth of flow at the stations. Such a procedure is usually carried out by trial and error.

In explaining this method it is convenient to refer the position of the water surface to a horizontal datum. In Fig. 10-6, the water-surface elevations above the datum at the two end sections are

$$Z_1 = S_0\,\Delta x + y_1 + z_2 \qquad (10\text{-}43)$$

and
$$Z_2 = y_2 + z_2 \tag{10-44}$$

The friction loss is

$$h_f = S_f \, \Delta x = \tfrac{1}{2}(S_1 + S_2) \, \Delta x \tag{10-45}$$

where the friction slope S_f is taken as the average of the slopes at the two end sections, or as $\bar{S}_f$.

Substituting the above expressions in Eq. (10-40), the following may be written:

$$Z_1 + \alpha_1 \frac{V_1^2}{2g} = Z_2 + \alpha_2 \frac{V_2^2}{2g} + h_f + h_e \tag{10-46}$$

where h_e is added for the eddy loss, which may be appreciable in nonprismatic channels. No rational method of evaluating eddy loss is available. The eddy loss depends mainly on the velocity head change and may be expressed as a part of it, or $k(\Delta\alpha V^2/2g)$ where k is a coefficient. For gradually converging and diverging reaches, $k = 0$ to 0.1 and 0.2, respectively. For abrupt expansions and contractions, k is about 0.5. For prismatic and regular channels, the eddy loss is practically zero, or $k = 0$. For convenience of computation, h_e may sometimes be considered part of the friction loss and Manning's n may be properly increased in computing h_f. Then, h_e is zero in the computation.

The total heads at two end sections are

$$H_1 = Z_1 + \alpha_1 \frac{V_1^2}{2g} \tag{10-47}$$

and
$$H_2 = Z_2 + \alpha_2 \frac{V_2^2}{2g} \tag{10-48}$$

Therefore, Eq. (10-46) becomes

$$H_1 = H_2 + h_f + h_e \tag{10-49}$$

This is the basic equation that defines the procedure of the standard step method.

The standard step method is best suited to computations for natural channels. A prismatic channel, however, will be used in the following example in order to simplify the illustration and to allow comparison with results obtained by the other methods that have been described. For application to natural channels, an example will be shown later (Art. 10-6).

Example 10-9. Compute the flow profile required in Example 10-1 by the standard step method. Assume that stations along the channel are fixed at the distances determined in the solution of Example 10-7. The elevation at the dam site is 600 m.s.l.

Solution. The step computations are arranged in tabular form, as shown in Table 10-6. Values in each column of the table are explained as follows:

Col. 1. Section identified by station number such as "station 1 + 55." The location of the stations is fixed at the distances determined in Example 10-7 in order to compare the procedure with that of the direct step method.

Col. 2. Water-surface elevation at the station. A trial value is first entered in this column; this will be verified or rejected on the basis of the computations made in the remaining columns of the table. For the first step, this elevation must be given or assumed. Since the elevation of the dam site is 600 m.s.l. and the height of the dam is 5 ft, the first entry is 605.00 m.s.l. When the trial value in the second step has been verified, it becomes the basis for the verification of the trial value in the next step, and so on.

Col. 3. Depth of flow in ft, corresponding to the water-surface elevation in col. 2. For instance, the depth of flow at station 1 + 55 is equal to water-surface elevation minus elevation at the dam site minus (distance from the dam site times bed slope), or $605.048 - 600.000 - 155 \times 0.0016 = 4.80$ ft.

Col. 4. Water area corresponding to y in col. 3

Col. 5. Mean velocity equal to the given discharge 400 cfs divided by the water area in col. 4

Col. 6. Velocity head in ft, corresponding to the velocity in col. 5

Col. 7. Total head computed by Eq. (10-47), equal to the sum of Z in col. 2 and the velocity head in col. 6

Col. 8. Hydraulic radius in ft, corresponding to y in col. 3

Col. 9. Four-thirds power of the hydraulic radius

Col. 10. Friction slope computed by Eq. (9-8), with $n = 0.025$, V from col. 5, and $R^{4/3}$ from col. 9

Col. 11. Average friction slope through the reach between the sections in each step, approximately equal to the arithmetic mean of the friction slope just computed in col. 10 and that of the previous step

Col. 12. Length of the reach between the sections, equal to the difference in station numbers between the stations

Col. 13. Friction loss in the reach, equal to the product of the values in cols. 11 and 12.

Col. 14. Eddy loss in the reach, equal to zero

Col. 15. Elevation of the total head in ft. This is computed by Eq. (10-49), that is, by adding the values of h_f and h_e in cols. 13 and 14 to the elevation at the lower end of the reach, which is found in col. 15 of the previous reach. If the value so obtained does not agree closely with that entered in col. 7, a new trial value of the water-surface elevation is assumed, and so on, until agreement is obtained. The value that leads to agreement is the correct water-surface elevation. The computation may then proceed to the next step. The computed flow profile is practically identical with that obtained by the graphical-integration method shown in Fig. 10-3.

10-5. Computation of a Family of Flow Profiles. In previous articles methods were described for determining a single flow profile. Frequently, several flow profiles, or a family of flow profiles, are desired for various conditions of stage and discharge. An example of this type of problem is the determination of the economical height of a dam, where the initial elevation is indeterminate and, hence, a number of flow profiles may have to be computed for the same discharge with different assumed

TABLE 10-6. COMPUTATION OF THE FLOW PROFILE FOR EXAMPLE 10-9 BY THE STANDARD STEP METHOD
$Q = 400$ cfs $n = 0.025$ $S_0 = 0.0016$ $\alpha = 1.10$ $h_e = 0$ $y_c = 2.22$ ft $y_n = 3.36$ ft

Station (1)	Z (2)	y (3)	A (4)	V (5)	$\alpha V^2/2g$ (6)	H (7)	R (8)	$R^{4/3}$ (9)	S_f (10)	$\bar{S}_f$ (11)	Δx (12)	h_f (13)	h_e (14)	H (15)
0 + 00	605.000	5.00	150.00	2.667	0.1217	605.122	3.54	5.40	0.000370	....	...	.....	...	605.122
1 + 55	605.048	4.80	142.08	2.819	0.1356	605.184	3.43	5.17	0.000433	0.000402	155	0.062	0	605.184
3 + 18	605.109	4.60	134.32	2.979	0.1517	605.261	3.31	4.92	0.000507	0.000470	163	0.077	0	605.261
4 + 91	605.186	4.40	126.72	3.156	0.1706	605.357	3.19	4.70	0.000598	0.000553	173	0.096	0	605.357
6 + 79	605.286	4.20	119.28	3.354	0.1925	605.479	3.08	4.50	0.000705	0.000652	188	0.122	0	605.479
8 + 91	605.426	4.00	112.00	3.572	0.2184	605.644	2.96	4.25	0.000850	0.000778	212	0.165	0	605.644
11 + 46	605.633	3.80	104.88	3.814	0.2490	605.882	2.84	4.02	0.001020	0.000935	255	0.238	0	605.882
13 + 04	605.786	3.70	101.38	3.948	0.2664	606.052	2.77	3.88	0.001132	0.001076	158	0.170	0	606.052
15 + 00	605.999	3.60	97.92	4.085	0.2856	606.285	2.71	3.78	0.001244	0.001188	196	0.233	0	606.285
16 + 23	606.146	3.55	96.21	4.158	0.2958	606.442	2.68	3.72	0.001310	0.001277	123	0.157	0	606.442
17 + 77	606.343	3.50	94.50	4.233	0.3067	606.650	2.65	3.66	0.001382	0.001346	154	0.208	0	606.650
18 + 98	606.507	3.47	93.48	4.278	0.3131	606.820	2.63	3.63	0.001427	0.001405	121	0.170	0	606.820
20 + 50	606.720	3.44	92.45	4.326	0.3202	607.040	2.61	3.59	0.001471	0.001449	152	0.220	0	607.040
21 + 87	606.919	3.42	91.80	4.357	0.3246	607.244	2.60	3.57	0.001500	0.001486	137	0.204	0	607.244
23 + 75	607.201	3.40	91.12	4.388	0.3292	607.530	2.59	3.55	0.001535	0.001518	188	0.286	0	607.530

CHAPTER 15

HYDRAULIC JUMP AND ITS USE AS ENERGY DISSIPATOR

15-1. The Hydraulic Jump. The hydraulic jump was first investigated experimentally by Bidone [1,2], an Italian, in 1818.[1] This led Bélanger [3] (1828) to distinguish between mild (subcritical) and steep (supercritical) slopes, since he had observed that in steep channels hydraulic jump is frequently produced by a barrier in originally uniform flow. Thereafter, abundant studies were made and the results were quoted by many writers. Outstanding contributors to our present knowledge about the hydraulic jump are Bresse (1860) [4], Darcy and Bazin (1865) [5], Ferriday and Merriman (1894) [6], Gibson (1913) [7], Kennison (1916) [8], Woodward and Riegel-Beebe (1917) [9], Koch and Carstanjen (1926) [10], Lindquist (1927) [11], Safranez (1927) [12], Einwachter (1933) [13,14], Smetana (1934) [15,16], Bakhmeteff and Matzke (1936) [17], Escande (1938) [18], Citrini (1939) [19], Nebbia (1940) [20], Kindsvater (1944) [21], Blaisdell (1948) [22], Forster and Skrinde (1950) [23], Rouse, Siao, and Nagaratnam (1958) [24], and many others.[2]

The theory of jump developed in early days is for horizontal or slightly inclined channels in which the weight of water in the jump has little effect upon the jump behavior and hence is ignored in the analysis. The results thus obtained, however, can be applied to most channels encountered in engineering problems. For channels of large slope, the weight effect of water in the jump may become so pronounced that it must be included in the analysis.

Practical applications of the hydraulic jump are many; it is used (1) to dissipate energy in water flowing over dams, weirs, and other hydraulic structures and thus prevent scouring downstream from the structures (Art. 15-8); (2) to recover head or raise the water level on the downstream side of a measuring flume and thus maintain high water level in the channel for irrigation or other water-distribution purposes; (3) to increase weight on an apron and thus reduce uplift pressure under a masonry structure by raising the water depth on the apron; (4) to increase the discharge of a sluice by holding back tailwater, since the effective head will be reduced if the tailwater is allowed to drown the jump;[1] (5) to indicate special flow conditions, such as the existence of supercritical flow or the presence of a control section so that a gaging station may be located; (6) to mix chemicals used for water purification, and so forth [28]; (7) to aerate water for city water supplies; and (8) to remove air pockets from water-supply lines and thus prevent air locking [29].

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[1] The experiment was made in Paris in 1818 and reported the following year in [1]. The hydraulic jump is also known as a *standing wave*. In French, it is called *le ressaut hydraulique*. In German, it is *der Wassersprung*. In honor of Bidone, the hydraulic jump in Italian is named *il salto di Bidone* (the jump of Bidone).

[2] For a comprehensive review of the studies on hydraulic jump, see [25]. For a mathematical treatment of the subject, see [26].

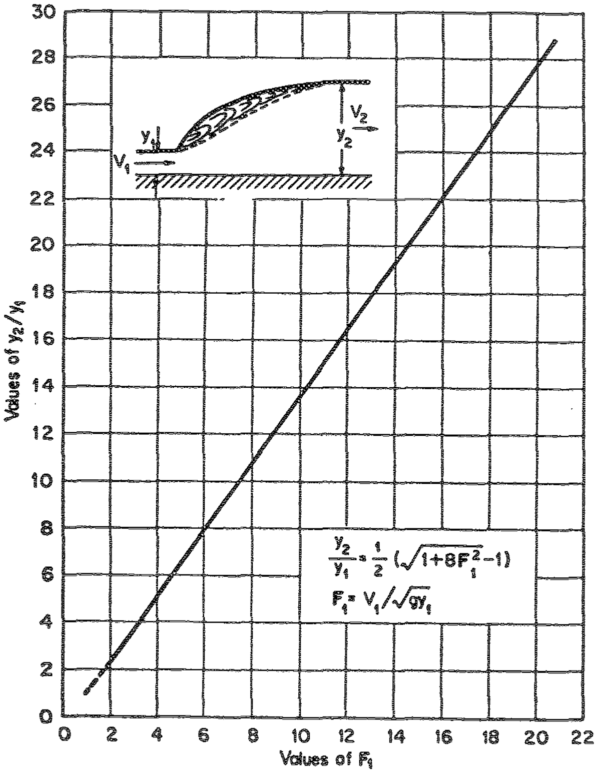

FIG. 15-1. Relation between F_1 and y_2/y_1 for a hydraulic jump in a horizontal rectangular channel.

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[1] This principle has been applied by Saugey [27] to an interesting device known as *fall increaser*. The device is intended to increase the effective head in a water-power plant during periods of flood by holding back tailwater from the outlet of the draft tube by a hydraulic jump.

15-2. Jump in Horizontal Rectangular Channels.[1] For supercritical flow in a horizontal rectangular channel, the energy of flow is dissipated through frictional resistance along the channel, resulting in a decrease in velocity and an increase in depth in the direction of flow. A hydraulic jump will form in the channel if the Froude number $\mathbf{F}_1$ of the flow, the flow depth y_1, and a downstream depth y_2 satisfy the equation

$$\frac{y_2}{y_1} = \tfrac{1}{2}\left(\sqrt{1 + 8\mathbf{F}_1^2} - 1\right) \qquad (3\text{-}21)$$

This equation may be represented by the curve in Fig. 15-1. This curve has been verified satisfactorily with many experimental data and will be found very useful in the analysis and design for hydraulic jumps.

15-3. Types of Jump. Hydraulic jumps on horizontal floor are of several distinct types. According to the studies of the U.S. Bureau of Reclamation [34,35], these types can be conveniently classified according to the Froude number $\mathbf{F}_1$ of the incoming flow (Fig. 15-2), as follows:

For $\mathbf{F}_1 = 1$, the flow is critical, and hence no jump can form.

For $\mathbf{F}_1 = 1$ to 1.7, the water surface shows undulations, and the jump is called an *undular jump*.

For $\mathbf{F}_1 = 1.7$ to 2.5, a series of small rollers develop on the surface of the jump, but the downstream water surface remains smooth. The velocity throughout is fairly uniform, and the energy loss is low. This jump may be called a *weak jump*.

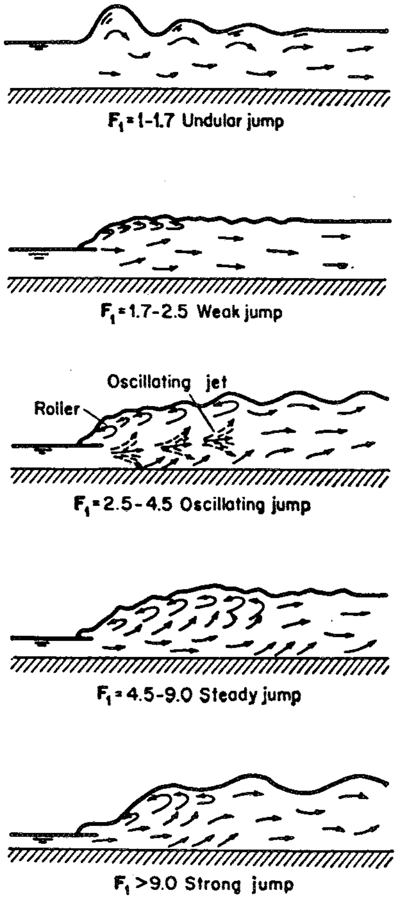
F_1 = 1-1.7 Undular jump
F_1 = 1.7-2.5 Weak jump
Oscillating jet
Roller
F_1 = 2.5-4.5 Oscillating jump
F_1 = 4.5-9.0 Steady jump
F_1 >9.0 Strong jump

FIG. 15-2. Various types of hydraulic jump.

For $\mathbf{F}_1 = 2.5$ to 4.5, there is an oscillating jet entering the jump bottom to surface and back again with no periodicity. Each oscillation produces a large wave of irregular period which, very commonly in canals, can travel for miles doing unlimited damage to earth banks and ripraps. This jump may be called an *oscillating jump*.

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[1] For hydraulic jumps in trapezoidal channels, see [30] and [31]. For jumps in closed conduits, see [29] and [32]. For a general treatment of nonrectangular channels, see [33].

For $F_1 = 4.5$ to 9.0, the downstream extremity of the surface roller and the point at which the high-velocity jet tends to leave the flow occur at practically the same vertical section. The action and position of this jump are least sensitive to variation in tailwater depth. The jump is well-balanced and the performance is at its best. The energy dissipation ranges from 45 to 70%. This jump may be called a *steady jump*.

For $F_1 = 9.0$ and larger, the high-velocity jet grabs intermittent slugs of water rolling down the front face of the jump, generating waves downstream, and a rough surface can prevail. The jump action is rough but effective since the energy dissipation may reach 85%. This jump may be called a *strong jump*.

It should be noted that the ranges of the Froude number given above for the various types of jump are not clear-cut but overlap to a certain extent depending on local conditions.

15-4. Basic Characteristics of the Jump. Several basic characteristics of the hydraulic jump in horizontal rectangular channels are to be discussed below:

Energy Loss. The loss of energy in the jump is equal to the difference in specific energies before and after the jump. It can be shown that the loss is

$$\Delta E = E_1 - E_2 = \frac{(y_2 - y_1)^3}{4y_1y_2} \tag{3-24}$$

The ratio $\Delta E/E_1$ is known as the *relative loss*.

Efficiency. The ratio of the specific energy after the jump to that before the jump is defined as the *efficiency of the jump*. It can be shown that the efficiency is

$$\frac{E_2}{E_1} = \frac{(8F_1^2 + 1)^{3/2} - 4F_1^2 + 1}{8F_1^2(2 + F_1^2)} \tag{15-1}$$

This equation indicates that the efficiency of a jump is a dimensionless function, depending only on the Froude number of the approaching flow. The relative loss is equal to $1 - E_2/E_1$; this also is a dimensionless function of F_1.

Height of Jump. The difference between the depths after and before the jump is the *height of the jump*, or $h_j = y_2 - y_1$. Expressing each term as a ratio with respect to the initial specific energy,

$$\frac{h_j}{E_1} = \frac{y_2}{E_1} - \frac{y_1}{E_1}$$

where h_j/E_1 is the *relative height*, y_1/E_1 is the *relative initial depth*, and

The experimental data on length of jump can be plotted conveniently with the Froude number $\mathbf{F}_1$ against a dimensionless ratio $L/(y_2 - y_1)$, L/y_1, or L/y_2. The plot of $\mathbf{F}_1$ vs. L/y_1 is probably the best, for the resulting curve can be best defined by the data. For practical purposes, however, the plot of $\mathbf{F}_1$ vs. L/y_2 is desirable, because the resulting curve shows regularity or a fairly flat portion for the range of well-established jumps. A curve of $\mathbf{F}_1$ vs. L/y_2 (Fig. 15-4) based on the experimental data of six test flumes has been prepared by the Bureau of Reclamation. In comparing this curve with the well-known Bakhmeteff-Matzke curve [17], pronounced disagreement was found. Investigation of the matter has led to the belief that this disagreement is due to the scale effect involved in Bakhmeteff and Matzke's experimental data. This scale effect means that the prototype action was not faithfully reproduced in the model. The curve shown in Fig. 15-4 was developed primarily for jumps occurring in rectangular channels. In the absence of adequate data, this curve may also be applied approximately to jumps formed in trapezoidal channels.

15-6. The Surface Profile. Knowledge of the surface profile of a jump is desirable in designing the freeboard for the retaining walls of the stilling basin where the jump takes place. It is important also for determining the pressure for use in structural design, because experiments have shown that the vertical pressure on the horizontal floor under a hydraulic jump is practically the same as would be indicated by the water-surface profile.

On the basis of their experimental data, Bakhmeteff and Matzke [17] have found that the surface profile of a hydraulic jump can be represented by dimensionless curves for various $\mathbf{F}_1$ values, as shown in Fig. 15-5. Moore [40] has developed similar curves for jumps below a free overfall. The profiles shown by Moore rise more rapidly at the beginning than do Bakhmeteff and Matzke's profiles. It is believed that this is because the nonhydrostatic-pressure distribution in the jump was not registered properly by the piezometric measurements for Bakhmeteff and Matzke's data. Furthermore, Moore's length of jump was about 20% longer than that shown by the Bakhmeteff-Matzke curves. Since the jump in the latter case was formed downstream from a regulating sluice, lack of agreement may be caused by a difference in the velocity profile of the shooting flow entering the jump.

15-7. Location of Jump. Hydraulic jump occurs in a supercritical flow when the depth changes abruptly to its sequent depth. Theoretically speaking, the jump will occur in a horizontal rectangular channel if the initial and sequent depths and the approaching Froude number satisfy Eq. (3-21). This theoretical condition is generally used to locate the position of a jump. For a closer estimate of the jump position, however, the length of the jump should be considered. The following will illustrate the location of a hydraulic jump in three typical cases (Fig. 15-6):

Case A shows the jump below a regulating sluice in a mild channel. The profiles AB and CD can easily be identified as of $M3$ and $M2$ type, respectively. The methods of computing these profiles are discussed in Chap. 10. The curve $A'B$ is a plot of the depth sequent to AB. By the position of F', the length of the jump can be estimated. By trial and error, a horizontal intercept between the curves $A'B$ and CD can be found equal to the length of jump. For instance, the horizontal distance EF is equal to the length of the jump corresponding to the depth y_2 at F. It becomes apparent that the jump will form between G and F, since the depth at F is sequent to the depth at G and the distance EF measures the length of the jump. It may be noted that, if the length of the jump were not taken into account in the analysis, the jump would have been considered to form at the upstream point F', resulting in an error represented by $F'F$. In case A, it can be seen that, by increasing the downstream water depth or raising the curve CD, the jump can be moved upstream. The downstream depth may be raised to such a height that the jump will eventually be drowned out in front of the sluice. Decreasing the downstream depth or lowering CD will move the jump downstream.

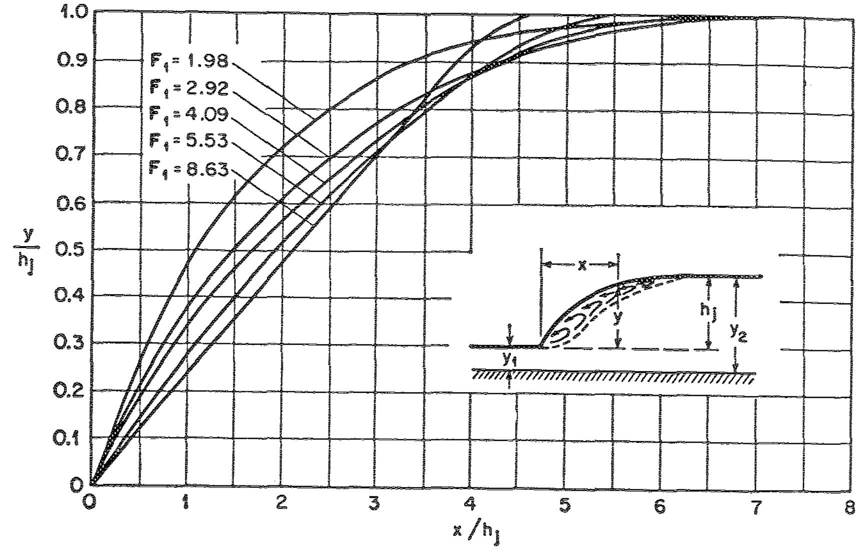

FIG. 15-5. Dimensionless surface profiles of hydraulic jumps in horizontal channels. (*Based on Bakhmeteff-Matzke data* [17].)

The above discussion applies also to the location of a jump formed at the foot of a weir or overflow spillway.

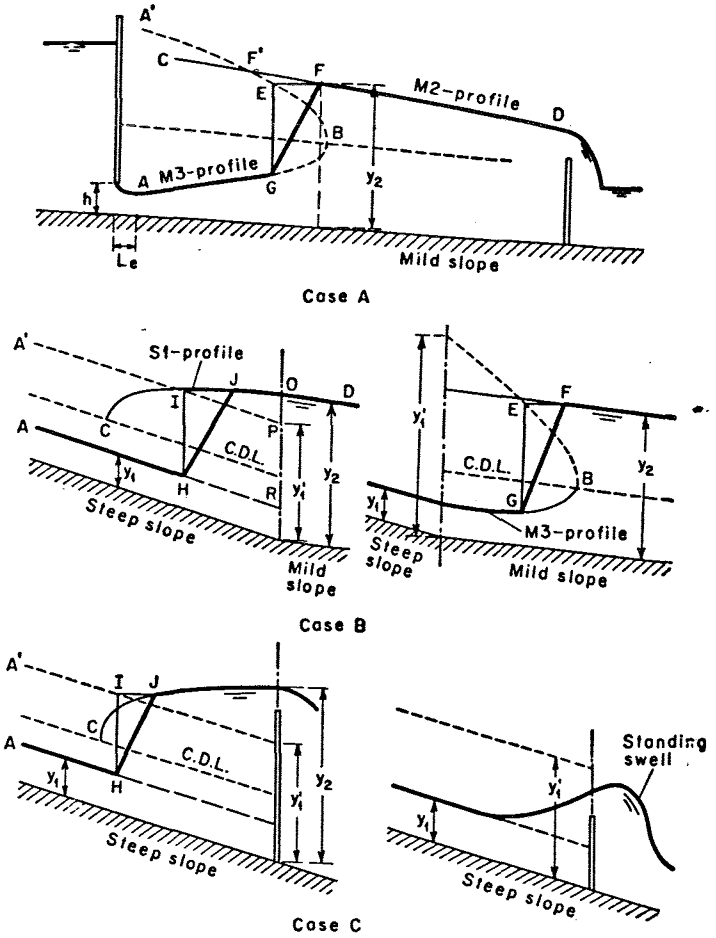

FIG. 15-6. Location of a hydraulic jump.

When there is a hydraulic jump below a sluice, the issuing flow from the sluice will form a jet that possesses a vena contracta. The distance L_e from the vena contracta to the sluice opening is usually short. Regarding this distance, there is a generally adopted rule which states that the vena contracta is located approximately at a distance h from the sluice opening.[1]

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[1] This rule was first used by Agroskin [41]. It is based on the assumptions that flow from a sluice corresponds to one-half the flow from a circular orifice and that the vena contracta in the flow from a circular orifice is located approximately at a distance of half an orifice diameter from the orifice. The rule of half orifice diameter was originated by Weisbach [42].

Case B shows the jump in a channel having a break in the bottom slope that changes from steep to mild. For simplicity, it is assumed that the flow is uniform in the channel except in the reach between the jump and the break. The jump may occur in either the steep channel or the mild channel, depending on whether the downstream depth y_2 is greater or less than the depth y_1' sequent to the upstream depth y_1. If the depth y_2 is greater than y_1', the jump will occur in the steep region. Then the surface curve OC is of $S1$ type. The line $A'P$ indicates the depth sequent to the line AR. Now, determine a horizontal intercept IJ between $A'P$ and CO that is equal to the length of a jump. It is apparent that a jump HJ will begin at the section containing I. If the depth y_2 is lowered approximately to less than y_1', the jump will start to move into the mild channel. In this case, the jump can be located as described in case A.

Case C shows the jump behind an overflow barrier. Theoretically, a jump will form if the depth at the barrier is greater than the depth y_1' sequent to the approaching supercritical depth y_1. The location of the jump is the same as that for case B if the jump occurs in the steep region. Increasing the height of the barrier will move the jump upstream. Decreasing the height will move the jump downstream. When the depth at the barrier is less than the sequent depth y_1', the barrier will be crossed by a *standing swell* in the form of a single undular surface rise which will not be followed by further undulations.

Example 15-1. Locate the hydraulic jump in Example 10-3 if the flow downstream from the jump is uniform.

Solution. From the given data, the specific-energy curve $E = y + \alpha V^2/2g$ and the specific-force curve $F = \beta Q^2/gA + \bar{z}A$ of the channel may be constructed as shown in Fig. 15-7. In computing the specific-force curve, the value of β may be estimated as 1.04 for $\alpha = 1.10$ [using Eqs. (2-6) and (2-7)].

Below the sluice, the $M3$ profile has been computed in Example 10-3, as shown by AGB in Fig. 15-8. Using the curves in Fig. 15-7 and following the method described in Art. 3-7, the curve of sequent depth $A'F'B$ corresponding to the curve of initial depth AGB can be determined.[1] The curve $A'F'B$ and the downstream flow profile CFD (equal to the normal-depth line in this example) intersect at F'. The initial depth of flow at F' is then found from the $M3$ profile to be 1.70 ft. The corresponding $\mathbf{F} = 1.52$ and, from Fig. 15-4, $L/y_2 = 3.6$. The length of the jump is therefore equal to $L = 3.6 \times 2.67 = 9.6$ ft. At this point there is an approximation involved, because the length of jump should be based on $\mathbf{F}$ at E (instead of F'), which, however, is as yet unknown.

In this example y_2 is equal to the normal depth of flow in the channel since the flow downstream is uniform. If the flow downstream is not uniform but gradually varied, then the depth at the intersection F' of the downstream profile with the curve $A'F'B$ should be taken as y_2. This is also an approximation, because the actual depth y_2 should be at F, the position of which is as yet unknown.

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[1] Owing to the difference between α and β, the computed critical depths indicated by the specific-energy and specific-force curves are not precisely identical. However, the discrepancy is so small that it can be ignored.

CHAPTER 16

FLOW IN CHANNELS OF NONLINEAR ALIGNMENT

16-1. Nature of the Flow. The presence of curves or bends in alignment is unavoidable in the design of open channels. Difficulties in design often arise because of the complexity of the flow around a curved path. The streamlines of the flow are not only curvilinear but also interwoven, resulting in spiral currents and cross waves. Furthermore, the centrifugal force acting on the flow around a bend produces a unique feature known as *superelevation*, that is, a rise in the water surface at the outer bank with an accompanying lowering at the inner bank. Also, the velocity distribution in the channel sections in the bend is very irregular and the coefficients α and β are usually far greater than unity.

In channels of nonlinear alignment, flows behave differently according to the state of the flow. Generally speaking, subcritical flow shows smooth water surface and slight superelevation, whereas supercritical flow exhibits characteristic cross-wave-disturbance patterns on the surface and thus exaggerates the superelevation.

In the study of subcritical flow, the spiral currents are of primary interest. These currents are essentially a friction phenomenon; hence their analysis requires use of the Reynolds number as a parameter.

In the study of supercritical flow, the formation of cross waves is of major concern. These waves represent the gravity effect of the free surface upon the flow; hence their analysis will rely on the use of Froude number as a basic parameter.

16-2. Spiral Flow. *Spiral flow* refers to movement of water particles along a helical path in the general direction of the flow. Thus, in addition to the major velocity component normal to the channel cross section, there are transverse velocity components on the cross section. These transverse components will create so-called *secondary flow* in the plane of the cross section.

Spiral flow in curved channels was first observed by Thomson [1] in 1876. Since then many studies have been made [2–7]. It is believed that this phenomenon is due mainly to (1) friction on the channel walls, which causes higher filamental velocities near the center of the channel than near the walls; (2) centrifugal force, which deflects the particles of water from straight-line motion; and (3) a vertical velocity distribution

From the experimental results obtained by Shukry for subcritical flow around a bend in a rectangular steel flume, the following are noted:

1. S_{xy} is comparatively high at low R of the approach flow, but decreases considerably by increasing R.
2. S_{xy} decreases gradually with the increase of the radius-width ratio r_c/b and attains practically its minimum magnitude (i.e., the curve effect approaches the least amount) at $r_c/b = 3.0$.
3. S_{xy} decreases as the depth-width ratio y/b increases.
4. S_{xy} increases as the deviation angle θ of the curve becomes large. For the range of $\theta/180°$ from 0.0 (straight channel) to 0.5, the increase in S_{xy} is nearly twice that for the range from 0.5 to 1.0.
5. The kinetic energy of the lateral currents in a curve is relatively small compared with the energy in the longitudinal currents and, consequently, plays only a minor part in the energy loss due to bend resistance.

Spiral flow occurs in natural rivers as well as in artificial channels [5]. However, it may not exist at all in a curved channel if the strength of the spiral motion is so weak that its effect is practically eliminated by the channel friction. This is the case in many natural rivers where the ratio of depth to width is small (hence, S_{xy} is small) and where the forces tending to produce spiral flow are overcome by the complicated forces resulting from bed and bank roughnesses [6,7].

16-3. Energy Loss. According to Müller [8], the energy line and flow profile in a uniform curved channel may be shown as in Fig. 16-1. Case 1 illustrates the *subcritical* flow in a curve between two tangent channels. Without the curve, flow would occur at the normal depth y_n corresponding to the particular discharge. With the curve, the energy line at the beginning A of the curve is raised by an amount h_f. A major part of this energy is dissipated over the length of the curve. The remaining part is carried over a distance L' in the downstream channel BB', which is required for the flow to revert to normal. The slope of the energy line between A and B' is greater than the bottom slope S_0, and the energy line meets the energy line for the normal flow at B'. In order to raise the energy line at the point A above the normal line, water must be backed up in the channel upstream from A. From the specific-energy curve of the flow in the channel, it can be shown that a rise of h_f in the energy line requires a corresponding rise in water surface by an amount Δy which is greater than h_f. This rise in water surface indicates that the presence of a bend in a channel of subcritical flow has a backwater effect similar to that of a weir or dam. For the computation of the backwater profile, the starting point may be set at A with a depth equal to $y_n + \Delta y$. The backwater profile is of the $M1$ type, which extends upstream from A and is asymptotic to the normal-depth line.

Case 2 illustrates *supercritical* flow in the curved channel. It can be seen that the energy line is dropped by h_f at B', corresponding to the amount of energy dissipated in the curve and the downstream channel BB'. The water surface is raised from the normal depth, starting at A, increasing to Δy at B', and then returning to normal afterwards. If the water surface rises above the critical-depth line, a hydraulic jump will be produced.

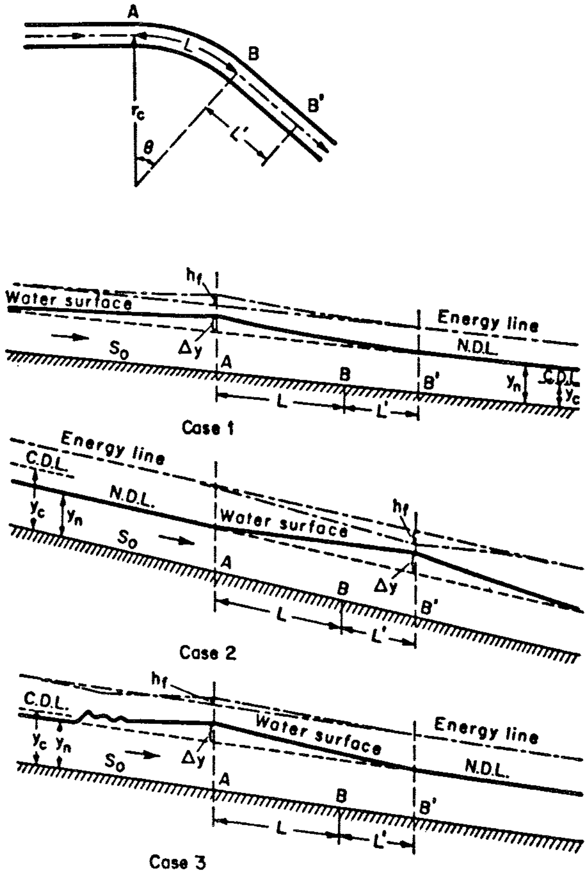

FIG. 16-1. Energy line and flow profile around a curve.

Case 3 illustrates *supercritical* flow when the normal depth is only slightly below the critical depth. The wavy surface in the upstream channel is due to the fact that the water surface is raised above the critical depth, so that an undular jump is produced.

The total energy loss due to curve resistance can be expressed in terms of the velocity head:

$$h_f = f_c \frac{V^2}{2g} \qquad (16\text{-}2)$$

where V is the mean velocity in the section and f_c is the coefficient of curve resistance. The coefficient f_c varies considerably with each of the parameters R of the approach flow, r_c/b, y/b, and $\theta/180°$. Families of curves for these parameters based on the experiments made by Shukry [2] (see [9] and [10] for other studies) are shown in Fig. 16-2. The curves can be used to determine approximately the value of f_c in smooth curved channels. For any given case, f_c is obtained by first fixing its value with respect to two variables and then adjusting with respect to the third and fourth variables.

Example 16-1. Given $\mathrm{R} = 55{,}500$, $r_c/b = 1.30$, $y/b = 0.8$, and $\theta/180° = 0.556$. Determine f_c.

Solution. First, keeping $y/b = 1.00$ and $\theta/180° = 0.50$, it is found that, for $\mathrm{R} = 55{,}500$ and $r_c/b = 1.30$, $f_c = 0.200$ (Fig. 16-2*c*). Then, keeping $r_c/b = 1.00$ and $\theta/180° = 0.50$, it is found that, for $\mathrm{R} = 55{,}500$ and $y/b = 1.00$, $f_c = 0.230$ and that, for $\mathrm{R} = 55{,}500$ and $y/b = 0.80$, $f_c = 0.275$ (Fig. 16-2*b*). Adjusting the condition of $y/b = 1.00$ to that of $y/b = 0.80$, the corrected coefficient $= 0.200 \times 0.275/0.230 = 0.239$. Now, keeping $y/b = 1.00$ and $r_c/b = 1.00$, it is found that, for $\mathrm{R} = 55{,}500$ and $\theta/180° = 0.556$, $f_c = 0.245$ (Fig. 16-2*a*). Similarly, adjusting the condition of $\theta/180° = 0.50$ to that of $\theta/180° = 0.556$, the finally corrected coefficient $= 0.239 \times 0.245/0.230 = 0.255$, say, 0.26.

The above procedure of interpolation is shown in Table 16-1.

TABLE 16-1. DETERMINATION OF THE COEFFICIENT OF CURVE RESISTANCE BY INTERPOLATION

Step	y/b	$\theta/180°$	R	r_c/b	f_c	Remarks
(1)	1.00	0.50	55,500	1.30	0.200	By Fig. 16-2*c*
(2)	1.00	0.50	55,500	1.00	0.230	By Fig. 16-2*b*
(3)	0.80	0.50	55,500	1.00	0.275	By Fig. 16-2*b*
(4)	0.80	0.50	55,500	1.30	0.239	By interpolation: (1)[(3)/(2)]
(5)	1.00	0.556	55,500	1.00	0.245	By Fig. 16-2*a*
(6)	0.80	0.556	55,500	1.30	0.255	By interpolation: (4)[(5)/(2)]

16-4. Superelevation. Studies on the superelevation in water surface around a curved channel are many [11–14]. From the experimental results obtained by Shukry (Fig. 16-3), it can be seen that the path of the thread of maximum velocity in a curved channel deviates from its normal course at a section upstream from the bend. At point *d*, the path almost touches the inside wall of the channel and the water surface attains its minimum level. Beyond point *d*, the path gradually moves outward until it crosses the center line of the flume at the section that passes lation constant can be multiplied by a correction factor equal to

$$\frac{\theta}{90^\circ} + \left(1 - \frac{\theta}{90^\circ}\right)\left(\frac{rV_{ss}}{C}\right)$$

where V_{ss} is the mean forward velocity in a straight channel.

Superelevation in curved channels may also be determined by less accurate but simpler formulas which are based on the application of Newton's second law of motion to the centrifugal action in the curve. Assuming that all filamental velocities in the bend are equal to the mean velocity V_s and that all streamlines have a radius of curvature r_c, the transverse water surface can be shown to be a straight line, and a simple formula for superelevation can be obtained:

$$\Delta h = \frac{V_s^2 b}{g r_c} \tag{16-11}$$

where b is the width of the channel.

Applying Newton's second law to each streamline and then integrating the whole channel section, Grashof [13] was able to show that the transverse surface profile is a logarithmic curve and that the superelevation is

$$\Delta h = 2.30 \frac{V_s^2}{g} \log \frac{r_o}{r_i} \tag{16-12}$$

Woodward [14,15] assumed that the velocity is zero at the banks and has a maximum value $V_{\max}$ at the center, varying in between according to a parabolic curve. Using Newton's second law, he obtained the following formula for superelevation:

$$\Delta h = \frac{V_{\max}^2}{g}\left[{}^{20}\!/_{3}\,\frac{r_c}{b} - 16\,\frac{r_c^3}{b^3} + \left(\frac{4r_c^2}{b^2} - 1\right)^2 \ln \frac{2r_c + b}{2r_c - b}\right] \tag{16-13}$$

Of the above three simple superelevation formulas, it has been found that Eq. (16-13) gives the best results, but none of them is more accurate than the free-vortex formula Eq. (16-8).

16-5. Cross Waves. Cross waves[1] are usually found in supercritical flow in channels of nonlinear alignment and channels with nonprismatic section (Chap. 17). These waves, forming a disturbance pattern that can persist for a considerable distance downstream, are caused by the turning effect of the curved walls, which does not act equally on all streamlines in the channel section. In a curved channel, the outer wall, which turns inward to the flow, will produce an oblique hydraulic jump and a corresponding positive disturbance line or positive wavefront[1] (Art. 15-17). The inner wall, which turns away from the flow, will develop a so-called *oblique expansion wave* and a negative disturbance line or negative wavefront[1] (Example 16-2). The disturbance lines thus produced by both outer and inner walls will be reflected back and forth between walls and will interfere with each other, resulting in a disturbance pattern of cross waves.

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[1] Cross waves in supercritical flow of water are analogous to the shock waves in supersonic flow of gases. This analogy was first noticed by Prandtl [16], Riabouchinsky [17], and von Kármán [18] and then investigated experimentally by Preiswerk [19] and others. These scientists, however, were interested primarily in applications of this principle to supersonic flow of gases. Later, comprehensive studies of the subject in hydraulic applications were made by Knapp, Ippen, and others [20–26]. To facilitate the analysis of shock wave in gases, Busemann [27] developed a graphical method known as the *method of characteristics*, which was later applied to hydraulic problems by Preiswerk, Ippen, and Knapp. For detailed descriptions of this method, the reader should refer to [23], [24], and [27].

Similarly, in any channel of nonlinear alignment, cross waves may be formed in various patterns, such as those shown in Fig. 16-4. It may be noted that the cross waves in Fig. 16-4*b* can be canceled, as shown in Fig. 16-4*c*, by making the line AA coincident with the first positive wavefront AA'. For large deflection angles of channel walls, the positive wavefront may be determined by the method developed in Art. 15-17 and the negative wavefront by the method to be described later in Example 16-2. For small deflection angles, such as the infinitesimal increment of the deflection angle in curved channels, the cross waves may be determined by the method described below.

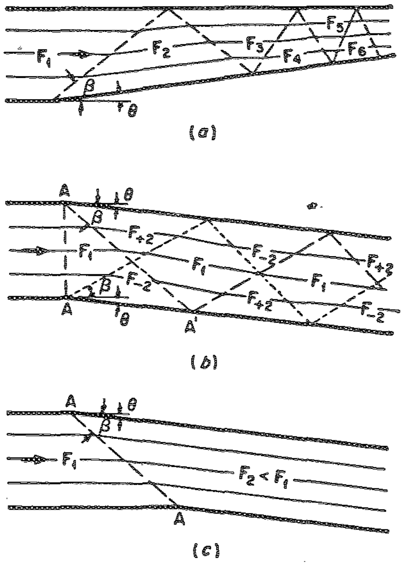

FIG. 16-4. Rectangular channels of nonprismatic alignment.

In studying the development of these methods it should be noted that, for small deflection angles and for large deflection angles that create negative wavefronts, the specific energy across the wavefront can be assumed constant since little energy dissipation is actually involved. For large deflection angles that create positive wavefronts, however, the specific energy cannot be assumed constant because the wavefront is built up to a substantial height and so the oblique hydraulic jump across the front will consume an appreciable amount of energy.

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[1] The disturbance line or wave angle created by an oblique hydraulic jump is considered positive in order to distinguish it from the negative disturbance line or wavefront due to an oblique expansion wave. The negative disturbance line actually only marks the beginning of a disturbance region; it is not a distinctly defined line like the positive one.

Considering the supercritical flow in a curved channel of constant width b and radius r_c (Fig. 16-5), the first small disturbance caused by the curvature of the outer wall starts at the beginning point A and is propagated along the line AB, which makes an initial wave angle β with the tangent extended beyond point A. In the meantime, the initial disturbance produced by the inner wall is propagated along the line $A'B$. The two propagation fronts meet at point B. Upstream from the bound-

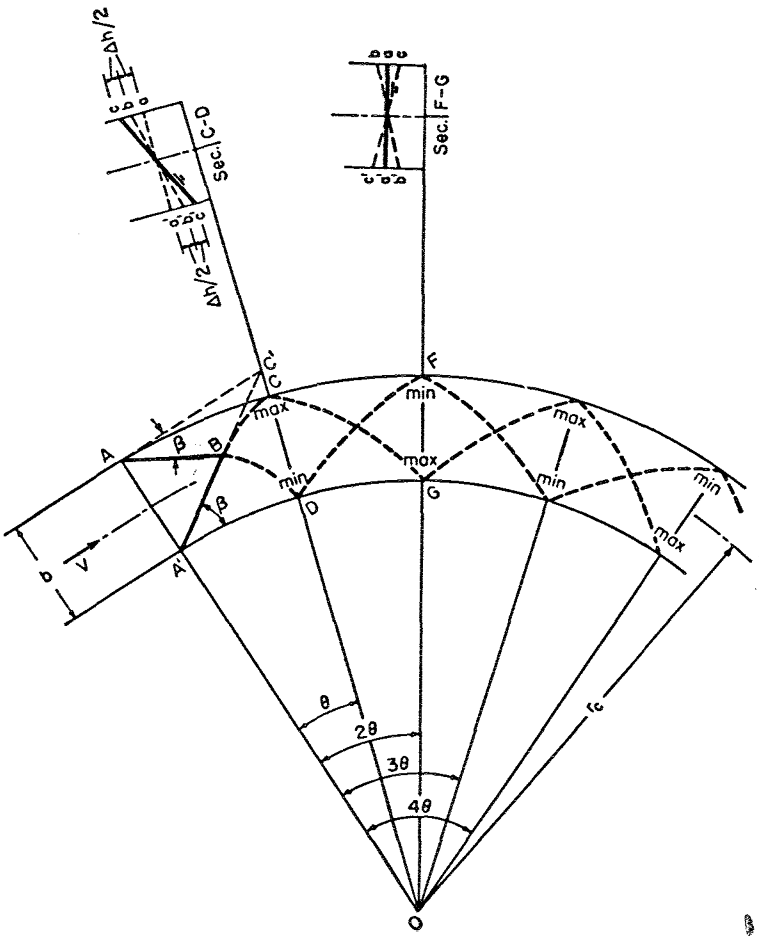

FIG. 16-5. Cross-wave pattern for supercritical flow in a curved channel.

ary line ABA', the flow is unaffected by the curve and thus continues to move in its original direction, that is, parallel to the upstream tangent. Beyond point B, the two wavefronts AB and $A'B$ affect each other and are no longer propagated in straight lines but in the curved paths BD and BC, respectively. The outer concave wall AC tends to deflect the flow, which would otherwise follow the tangential direction. Consequently, the water surface is raised higher and higher around the outer wall up to a maximum at C. After C the effect of the inner wall, which is to lower the water surface, begins to operate. Thus, the water surface along the outer wall starts to drop. On the inner convex side, the flow tends to depart from the wall, and the water surface is depressed lower and lower around $A'D$ until the point D is reached. After D the effect of the outer wall comes into play and the surface begins to rise again. The reflection of disturbance waves from the outer and inner walls will not come to equilibrium or stop when they meet near the center of the channel. They will continue to be reflected back and forth across the channel, causing the surface profiles along the walls to have a series of maxima and minima of surface elevation, approximately at angles θ, 3θ, 5θ, . . . , from the beginning of the curve. The angle θ marks half the wavelength of the disturbance pattern. For practical purposes, it may be assumed that the points of maximum and minimum for each phase angle θ occur on the same radial line, such as OC for the first maximum and minimum.[1] By geometry (Fig. 16-5), the central angle to the first maximum can be shown to be

$$\theta = \tan^{-1} \frac{2b}{(2r_c + b) \tan \beta} \tag{16-14}$$

where the wave angle β is approximately $\sin^{-1} (\sqrt{gy}/V)$, as will be shown below by Eq. (16-15).

In order to simplify the computation of the water-surface elevation[2] the following major assumptions are to be made: (1) two-dimensional flow, (2) constant velocity across the cross section, (3) horizontal channel, (4) frictionless flow, and (5) vertical channel walls. Assumptions 3 and 4 do not exclude the application of the results to sloping channels if the slope compensates for the friction. ie - normal flow

For the curved outer wall, the changes in the angle of deflection and in the depth are gradual and small and, thus, can be represented by $d\theta$ and

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[1] Actually the locations of the first maximum and minimum do not occur exactly on the radial line OC but slightly on the left side and right side of OC, respectively (Fig. 16-5). However, the error involved in the assumption is negligible. This assumption applies also to the locations of the subsequent maxima and minima.

[2] A detailed study of the complete surface contours is possible by means of the method of characteristics [23].

dy, respectively. Assuming $y_1 = y_2 = y$ for an oblique hydraulic jump of infinitesimal height dy, Eq. (15-21) becomes[1]

$$\mathbf{F}_1 = \mathbf{F} = \frac{1}{\sin\beta} = \frac{V}{\sqrt{gy}} \tag{16-15}$$

Substituting y for y_1 and $y + dy$ for y_2, Eq. (15-23) may be reduced to

$$dy = \frac{y \sec^2\beta \tan\theta}{\tan\beta - \tan\theta} \tag{16-16}$$

For a small angle, $\tan\theta$ may be replaced by $d\theta$, and $\tan\theta$ in the denominator is negligible compared with $\tan\beta$. Thus, Eq. (16-16) becomes

$$dy = \frac{y}{\sin\beta\cos\beta}\,d\theta \tag{16-17}$$

By combining Eqs. (16-15) and (16-17), the following is obtained:

$$dy = \frac{V^2}{g}\tan\beta\,d\theta \tag{16-18}$$

According to assumption 4 above, the specific energy may be considered constant. Since $E = y + V^2/2g$, $V = \sqrt{2g(E - y)}$. Substituting this expression for V in Eqs. (16-15) and (16-18) and then eliminating β by Eq. (16-15), the following is obtained:

$$\frac{dy}{d\theta} = \frac{2(E - y)\sqrt{y}}{\sqrt{2E - 3y}} \tag{16-19}$$

The exact solution of Eq. (16-19) for θ gives

$$\theta = \sqrt{3}\tan^{-1}\sqrt{\frac{3y}{2E - 3y}} - \tan^{-1}\frac{1}{\sqrt{3}}\sqrt{\frac{3y}{2E - 3y}} + \text{const} \qquad \text{(16-20)*}$$

Since $2E = y(2 + \mathbf{F}^2)$ (why?), Eq. (16-20) may be written

$$\theta = \sqrt{3}\tan^{-1}\frac{\sqrt{3}}{\sqrt{\mathbf{F}^2 - 1}} - \tan^{-1}\frac{1}{\sqrt{\mathbf{F}^2 - 1}} + \text{const} \tag{16-21}$$

This equation can be used to calculate the change in depth along the walls at the beginning of the curve. The integration constant can be determined by the condition that, for $\theta = 0$, the depth y is the initial depth y_1. Equation (16-21) is, however, involved and inconvenient for practical usage, even with the aid of a graphical chart such as that developed by Ippen [23]. According to Knapp and Ippen [21–24], adequate results may be obtained by the much simpler equation

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[1] This equation is believed to be true only for supercritical flow in wide rectangular channels. For broad applications, Engelund and Munch-Petersen [28] have developed a generalized equation, which has been found in good agreement with experimental data covering considerable variation of the Froude number (even for F equal to or slightly less than unity) and of the depth-width ratio of the channel.

* The mathematical derivation of the equation of this form was first made by von Kármán [18].

$$y = \frac{V^2}{g} \sin^2 \left(\beta + \frac{\theta}{2} \right) \tag{16-22}$$

This equation was developed as a result of actual flow measurements.[1] In using Eqs. (16-21) and (16-22), the angle θ is positive for depths along the outer wall and negative for depths along the inner wall. The depth at the first maximum heights of the cross-wave disturbance may be obtained by using the value of θ computed by Eq. (16-14).

In cross section *CD* (Fig. 16-5), where the first maximum height of the cross wave occurs at the outer wall, the line *a'a* represents the theoretical position of the water surface in the channel cross section if the channel were straight, line *b'b* the theoretical position of the water surface on the curve if the flow were subcritical, and line *c'c* the actual water surface on the curve when the flow is supercritical. It is evident that *b* is higher than *a* and *b'* is lower than *a'* by an amount equal to half the superelevation, or $\Delta h/2$. By Eq. (16-11), this amount is $V^2 b/2r_c g$. According to the experimental investigations made by Ippen and Knapp [21,23], *c* is higher than *a* by an amount equal to about Δh, or higher than *b* by about $\Delta h/2$. The position *c'* is lower than *b'* by about $\Delta h/2$. Similar conditions exist in other cross sections where maximum height of water surface occurs at the outer wall.

In cross section *FG*, where the minimum cross-wave height occurs at the outer wall, the actual water surface is identical with *a'a* because the effect of the disturbance cross wave is practically offset by the superelevation. Similarly, in other cross sections where the wave height is minimum at the outer wall, the water surface assumes a position as if the flow were in a straight channel.

The distance *AC* along the wall represents a half wavelength, which sustains a central angle θ. This length may be approximated by *AC'* or $b/\tan\beta$ by assuming that the angle $AC'A' = \beta$. Thus, the wavelength is $2b/\tan\beta$.

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[1] On the basis of assumption 4 and, hence, of the conservation of energy, the velocity in any streamline must change as the depth changes, since $E = y + V^2/2g$ must be constant. The flow around the outer wall, being the deepest, should be the slowest. Actual measurements, however, indicate on the contrary that the velocity around the outer wall remains constant or even increases slightly, whereas along the inner wall the velocity decreases. It is believed that this disparity is due to varying effects of the channel friction, which actually render assumption 4 invalid. Consequently, it is reasoned that a constant velocity may be assumed. Equation (16-22) was, therefore, derived from the assumption of constant velocity, which replaces assumption 4. Since the author cannot discern here the rigor of the original mathematical derivation, he would rather consider Eq. (16-22) to be empirical.

From the above discussion, it may be concluded that the disturbance wave pattern, which oscillates about the plane represented by $b'b$, has a wavelength of $2b/\tan\beta$ and an amplitude of $V^2b/2r_cg$. This finding can be used to estimate roughly the supercritical-flow profile in simple curved channels.

For disturbance continuing into the downstream tangent channel, the primary wavelength is still $2b/\tan\beta$. As the curvature suddenly changes to straight alignment, a new disturbance pattern, starting with a maximum height on the outer wall at the point of tangency, has a wavelength and magnitude equal to that of the original disturbance pattern developed in the curve. The resulting disturbance pattern in the tangent is the sum of the original and new patterns. The new disturbance pattern may, therefore, be eliminated by adopting curve lengths of 2θ, 4θ, . . . , which will have a minimum wave height at the outer wall at the end of the curve, just enough to cancel out the newly created maximum wave height.

Example 16-2. Describe the characteristics of an oblique expansion wave in channels of nonlinear alignment.

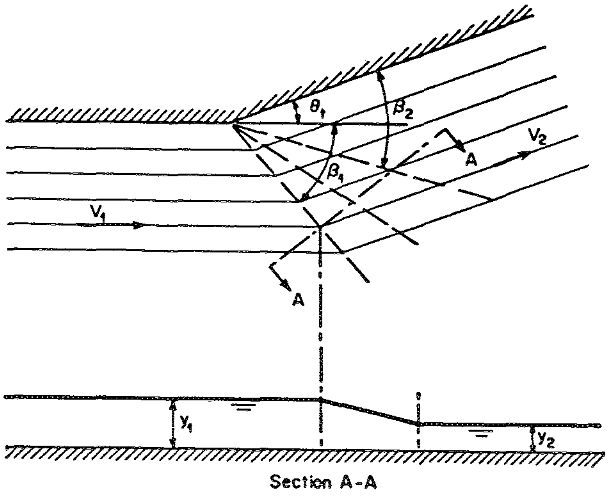

FIG. 16-6. Oblique expansion wave.

Solution. The oblique expansion wave occurs when the channel wall is turned outward from the flow at an angle θ_t (Fig. 16-6). The depth of flow decreases in the fan-shaped region of the wave disturbance delineated by wave angles β_1 and β_2, measured with reference to the initial and final flow directions, respectively. By Eq. (16-15),

$$\beta_1 = \sin^{-1}\frac{1}{F_1} \qquad \text{and} \qquad \beta_2 = \sin^{-1}\frac{1}{F_2} \tag{16-23}$$

17.52. The origin of the coordinates for this equation is on the water surface at a point equidistant from the banks. In using this equation the following remarks should be noted:

1. For r_o less than 40 times the square root of the water area, no further deepening of the channel seems to result from the increased curvature; hence, in such cases, the value of r_o used in the equation should be $40\sqrt{A}$. Consequently, bends are constructive and stable when r_o is greater than $40\sqrt{A}$, whereas sharper bends are destructive, tending to shift the channel.
2. For r_o greater than about $110\sqrt{A}$, the equation becomes invalid.

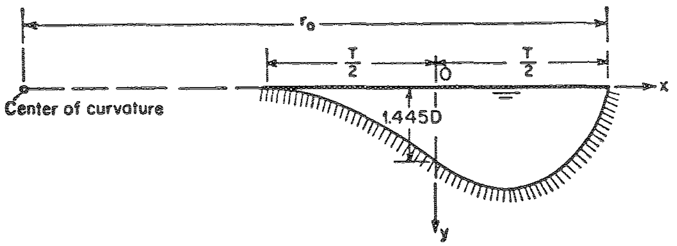

FIG. 16-7. Empirical channel cross section at river bend.

3. The equation may be applied to curved channels not occupying the entire width of the waterway or to those at the river entrance created by a single curved jetty. In such cases, $K = 26.28$ and the value of y thus computed should be increased 14%.
4. On a crossover bar when the channel is neither on a curve nor in a straight reach, the maximum depth is about 14.5% less than the computed value.
5. The equation generally gives a width of channel at hydraulic depth about 20% greater than the actual width.
6. The equation is very approximate; nevertheless, in the absence of a better method for determining shape of cross section at a river bend or at a bend in a dredged canal, it will give satisfactory answers to many practical problems.

16-7. Design Considerations for Supercritical Flow. The major issue in the design of curved channels for supercritical flow is to eliminate or reduce the superelevation and cross-wave disturbance pattern. Knapp [24] has suggested the following methods to achieve this objective:

A. Banking. By banking is meant use of a bottom cross slope that will supply a lateral force to counteract the centrifugal action of the flow. The required cross slope S_t can be computed by equating the gravity component along the cross slope to the centrifugal force determined by the radius r of the curve and the velocity V; that is,

$$S_t = \frac{V^2}{gr} \tag{16-26}$$

In order to avoid abrupt changes in flow condition, banking should be introduced gradually from zero to its full amount, starting at both ends of the curve. It should be noted that the slope thus computed is good only for the velocity given. When the velocity changes, the computed banking will be less effective under new flow conditions. Hence, banking is most suitable in channels which ordinarily operate at or near the designed flow condition. Factors against banking include costly excavation and possible silting or erosion along the inner wall during low flows.

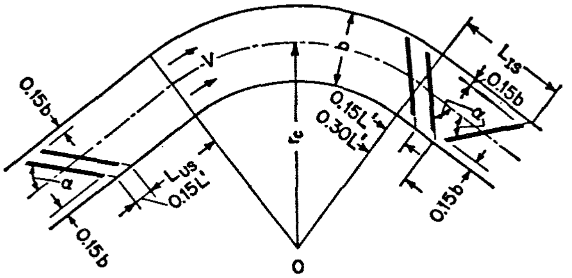

FIG. 16-8. Plan of sill installation in a curved channel. (*After R. T. Knapp* [24].)

B. Multiple Curved Vanes. The superelevation and disturbance pattern can be reduced by concentric vanes which divide the channel width into a series of narrow curved channels. This method is not generally practical, and it becomes impossible in channels carrying debris of sizes larger than the subdivided width.

C. Easement Curves. The disturbance in a simple curved channel may be reduced by employing a compound curve. The best design is a simple curve of radius r_c preceded and followed by a section of another simple curve, whose length is $b/\tan \beta$ and whose radius is $2r_c$. The resulting compound curve will offer a very desirable solution for most curved channels of supercritical flow. Other types of easement curve, such as a spiral transition curve, will increase appreciably the cost of design and construction with only slight improvement in flow characteristics.

D. Diagonal Sills. Diagonal sills installed on the bottom of the channel near the ends of the curve will produce an effect corresponding to that of an easement curve. An effective layout of the sills, shown in Fig. 16-8, has been developed experimentally. The optimum sill angle

α is 30°. The distance L_{US} may be estimated by

$$L_{US} = \frac{Kb}{\tan \beta} \tag{16-27}$$

where K is a coefficient. For normal cases, the value of K lies between 0.9 and 1.15. In accordance with the figure the length L_{IS} is

$$L_{IS} = 0.30L' + \frac{b}{\tan \alpha} \tag{16-28}$$

where $L' = b/\tan \beta$, that is the half wavelength of a channel disturbance.

This method can be used as a remedial measure in existing channels which have been designed as simple circular curves or other unsatisfactory forms. The major disadvantages of this method are high maintenance cost, pronounced disturbance at low flows, and possible cavitation at extreme high-velocity flows. The pronounced disturbance at low flows may cause uneasiness and distrust in an unexperienced observer as he may not be easily convinced that the disturbance will be greatly reduced at high design flows.

PROBLEMS

16-1. A 120° bend with $b = 10$ in. and $r_c = 15$ in. is designed to connect two straight channels of the same width, carrying 1.3 cfs at a normal depth of 7 in. Determine the bend loss.

16-2. Determine the specific energy at the section containing maximum surface depression in the bend given in the preceding problem. Assume $\alpha = 1.0$ and $n = 0.01$.

16-3. Compute the water-surface profile in the cross section containing maximum surface depression, as described in the preceding problem, using the law of free vortex. What is the superelevation?

16-4. Determine the superelevation in the preceding problem by (*a*) Eq. (16-11), (*b*) Eq. (16-12), and (*c*) Eq. (16-13).

16-5. Verify Eq. (16-14).

16-6. Determine the approximate flow profile in the curved channel given in Prob. 16-1 if the approach channel carries a supercritical flow at a depth of 4.5 in.

16-7. Design the curved channel for a rectangular flume to turn an angle of 50° with a radius of 250 ft. The flume is 12 ft wide and built of smooth concrete. The design discharge is 350 cfs at a slope of 1%.

16-8. Using Eq. (16-25), compute the channel cross section at a bend in the Mississippi River, where $A = 148{,}010$ ft², $T = 2{,}340$ ft, and $r_c = 18{,}300$ ft. Compare the result with the actual cross section, which is as follows:

x	y	x	y	x	y
1,170	0.00	200	97.00	−600	44.00
1,000	36.00	0	74.00	−800	38.00
800	88.00	−200	65.00	−1,000	24.00
600	108.00	−400	55.00	−1,170	0.00
400	111.00				